Spring 2003 Math 308/501–502
6 Theory of Higher-Order Linear ODEs
6.3 Method of Undetermined Coeffs
Wed, 01/Oct ©2003, Art Belmonte

Summary

A *nonhomogeneous* linear ODE of order n with real constant coefficients has the form $L[y] = \sum_{k=0}^{n} a_k y^{(k)} = f$. Here t (or x) is the independent variable and the nonhomogeneity $f \neq 0$ is known as the "forcing function;" it may consist of one or several terms.

The associated *homogeneous* ODE is $L[y] = \sum_{k=0}^{n} a_k y^{(k)} = 0$. A general solution y_h of this is obtained via §6.2 methods.

General solution of the nonhomogeneous equation

If y_p is a particular solution of the nonhomogeneous equation and y_h is a general solution to the associated homogeneous equation, then a general solution of the nonhomogeneous equation is given by $y = y_p + y_h$.

Superposition Principle

For $k = 1, 2, \ldots, M$, let y_{p_k} be a solution of $L[y] = f_k$. Then for any constants $c_1, \ldots, c_M$, the function $y_p = \sum_{k=1}^{M} c_k y_{p_k}$ solves the differential equation $L[y] = \sum_{k=1}^{M} c_k f_k$. (This follows immediately from the fact that L is a linear differential operator.)

Method of Undetermined Coefficients

If the forcing function $f(t) = \sum_{k=1}^{M} f_k(t)$ is a sum of products of real polynomials, sines, cosines, and/or exponentials, then the following method produces a **general solution** of $L[y] = f(t)$. Each $f_k(t)$ *must* have the form

$$e^{\alpha t}\left(p_u(t)\cos\beta t + q_v(t)\sin\beta t\right)$$

where p_u and q_v are polynomials of degrees u and v, whereas α and β are real constants. Now α or β may be 0, p_u or q_v may be constants, and these may all vary with the index k. Often, M is 1; i.e., there is a single term in the forcing function.

1. *First* obtain a general solution y_h of $L[y] = 0$.
2. For each k, determine a particular solution y_{p_k} of $L[y] = f_k(t)$, as follows. (If $M = 1$, just use y_p for y_{p_1}.)
   (a) Form $y_{p_k} = t^s e^{\alpha t}\left(P_N(t)\cos\beta t + Q_N(t)\sin\beta t\right)$, where $N = \max(u, v)$ and s is the *smallest* nonnegative integer so that no term of the particular solution y_{p_k} is a solution of the corresponding homogeneous equation $L[y] = 0$. Here P_N and Q_N are general polynomials of degree N which have **undetermined coefficients**; i.e., the coefficients are *symbolic* at this stage. From y_{p_k}, compute derivatives $y_{p_k}^{(j)}$, $j = 1, 2, \ldots, n$, then *substitute* into $L[y] = f_k(t)$.
   (b) Collect and equate like terms. This results in a linear system of equations. Solve for the undetermined coefficents. Now you know y_{p_k}.
3. Let $y_p = \sum_{k=1}^{n} y_{p_k}$. The **general solution** of $L[y] = f(t)$, via superposition, is $y = y_p + y_h$.

By hand or with MATLAB, you're a winner!

The hand work involved in computing derivatives, substitution, collecting terms, and solving linear systems can be tedious and error-prone. It lends itself quite well, however, to machine power, as you'll observe in the ***MATLAB Examples***.

You may have felt that the formulation for y_p given above is overly complicated. Take solace in the fact that under that stated formulation there is only *one* case (the general case). What's more, there is *no* guesswork as to the correct *form* of y_p. With practice, you'll get it right every time!

Note that in all examples (hand or MATLAB), we carry the work through to a full general solution (or to the unique solution of an initial value problem). Please do this in your homework problems as well.

Hand Example

In general, when dealing with ODEs of order 3 or higher, we use MATLAB! Hand computations are too lengthy. Here's one done by hand for grins. You'll see what I mean. (Please see the Section 4.5 lecture handout for two more hand examples.)

337/1

Find a general solution of $y''' - 2y'' - 5y' + 6y = e^x + x^2$.

Solution

Even with a TI-89 to help, this is cumbersome!

1. The homogeneous equation $y''' - 2y'' - 5y' + 6y = 0$ has characteristic equation $r^3 - 2r^2 - 5r + 6 = 0$, with roots $r = -2, 1, 3$. Hence $y_h = c_1e^{-2x} + c_2e^x + c_3e^{3x}$.
2. Let $y_p = x \cdot ae^x + bx^2 + cx + d$. Its derivatives are $y_p' = a(x+1)e^x + 2bx + c$, $y_p'' = a(x+2)e^x + 2b$, and $y_p''' = a(x+3)e^x$. Substitute into the differential equation.
   $$a(x+3)e^x - 2(a(x+2)e^x + 2b) - 5(a(x+1)e^x + 2bx + c)\ldots$$
   $$+\ 6(axe^x + bx^2 + cx + d) = e^x + x^2.$$
   After collecting, we have $-6ae^x + 6bx^2 + (6c - 10b)x - 4b - 5c + 6d = e^x + x^2$.
3. Equating coefficients yields a system of four equations: $-6a = 1,\ 6b = 1,\ 6c - 10b = 0,\ -4b - 5c + 6d = 0$. Solving these gives $a = -\frac{1}{6},\ b = \frac{1}{6}\ c = \frac{5}{18},\ d = \frac{37}{108}$. Thus $y_p = -\frac{1}{6}xe^x + \frac{1}{6}x^2 + \frac{5}{18}x + \frac{37}{108}$ is our particular solution.
4. A general solution (verified by **dsolve**) is $y = y_p + y_h$
   $$= -\frac{1}{6}xe^x + \frac{1}{6}x^2 + \frac{5}{18}x + \frac{37}{108} + c_1e^{-2x} + c_2e^x + c_3e^{3x}.$$

MATLAB Examples

Bring the requisite machine power to bear upon the problem!

337/1 [revisited]

Find a general solution of $y''' - 2y'' - 5y' + 6y = e^x + x^2$.

Solution

```
%
% NSS4-337/1
%
syms a b c d c1 c2 c3 r x
y = sym(’y(x)’);
p = poly2sym([1 -2 -5 6], r); pretty(p)

                               3      2
                              r  - 2 r  - 5 r + 6
r = solve(p)

r =

[  1]
[ -2]
[  3]

yh = c1*exp(x) + c2*exp(-2*x) + c3*exp(3*x);
%
L = diff(y,x,3) - 2*diff(y,x,2) - 5*diff(y,x) + 6*y;
pretty(L)
%
%
               / 3      \     / 2      \
               |d       |     |d       |     /d       \
               |--- y(x)| - 2 |--- y(x)| - 5 |-- y(x)| + 6 y(x)
               |  3     |     |  2     |     \dx      /
               \dx      /     \dx      /
yp = x * a*exp(x)  +  b*x^2 + c*x + d;
eq0 = subs(L - (exp(x) + x^2), y, yp);
eq0 = collect(eq0, exp(x));
eq0 = collect(eq0, x)

eq0 =

(6*b-1)*x^2+(6*c-10*b)*x+(-1-6*a)*exp(x)-4*b+6*d-5*c

[a b c d] = solve(6*b-1, 6*c-10*b, -1-6*a, -4*b+6*d-5*c)

a =

-1/6


b =

1/6


c =

5/18


d =

37/108

%
yp = subs(yp); pretty(yp)

                                          2            37
                    - 1/6 x exp(x) + 1/6 x  + 5/18 x + ---
                                                       108
check = subs(L, y, yp)

check =

exp(x)+x^2

y = yp + yh; pretty(y)

                      2            37
- 1/6 x exp(x) + 1/6 x  + 5/18 x + --- + c1 exp(x) + c2 exp(-2 x)
                                   108
         + c3 exp(3 x)
%
sol = dsolve(’D3y - 2*D2y - 5*Dy + 6*y = exp(x) + x^2’, ’x’);
sol = simplify(sol);
pretty(sol)

                      2            37
- 1/6 x exp(x) + 1/6 x  + 5/18 x + --- - 1/36 exp(x) + C1 exp(x)
                                   108
         + C2 exp(3 x) + C3 exp(-2 x)
% This is equivalent to our answer
% when we set C1 - 1/36 = c1.
%
echo off; diary off
```

337/31

Solve the IVP $y''' + 2y'' - 9y' - 18y = -18x^2 - 18x + 22$; $y(0) = -2, \quad y'(0) = -8, \quad y''(0) = -12.$

(Please turn the page for the solution.)

Solution

```
%
% NSS4-337/31
%
syms a b c c1 c2 c3 r x
y = sym('y(x)');
p = poly2sym([1 2 -9 -18], r); pretty(p)

   3      2
  r  + 2 r  - 9 r - 18
r = solve(p)

r =

[ -2]
[ -3]
[  3]

yf = [exp(-2*x), exp(-3*x), exp(3*x)];
%
L = diff(y,x,3) + 2*diff(y,x,2) - 9*diff(y,x) - 18*y;
pretty(L)

  / 3      \     / 2      \
  |d       |     |d       |     /d      \
  |--- y(x)| + 2 |--- y(x)| - 9 |-- y(x)| - 18 y(x)
  |  3     |     |  2     |     \dx     /
  \dx      /     \dx      /
yp = a*x^2 + b*x + c;
eq0 = subs(L - (-18*x^2 - 18*x + 22), y, yp);
eq0 = collect(eq0, x)

eq0 =

(18-18*a)*x^2+(18-18*b-18*a)*x-22+4*a-18*c-9*b

[a b c] = solve(18-18*a, 18-18*b-18*a, ...
    -22+4*a-18*c-9*b)

a =

1


b =

0


c =

-1

%
yp = subs(yp); pretty(yp)

   2
  x  - 1
% We're done with a,b,c (for now)...
check = subs(L, y, yp)

check =

-18*x^2-18*x+22

%
v = [yf yp];
%
M = wron(v, x); M = subs(M, x, 0) % PUSH WRON!
M =
     1     1     1    -1
    -2    -3     3     0
     4     9     9     2
    -8   -27    27     0
% Redefining a, b, c to resolve ICs.
a = M(1:3, 4)
a =
    -1
     0
     2
M = M(1:3, 1:3)
M =
     1     1     1
    -2    -3     3
     4     9     9
b = sym([-2; -8; -12])

b =

[  -2]
[  -8]
[ -12]

c = M\(b-a)

c =

[  1]
[  0]
[ -2]

y = yp + yf*c; pretty(y)

   2
  x  - 1 + exp(-2 x) - 2 exp(3 x)
% Beauty, eh?
%
sol = dsolve(...
    'D3y + 2*D2y - 9*Dy - 18*y = -18*x^2 -18*x + 22', ...
    'y(0)=-2', 'Dy(0)=-8', 'D2y(0)=-12', 'x');
pretty(sol)

   2
  x  - 1 + exp(-2 x) - 2 exp(3 x)
%
echo off; diary off
```

337/29

Find a general solution of $z''' - 2z'' + z' = x - e^x$.

Solution

```
%
% NSS4-337/29
%
syms a b c c1 c2 c3 r x k1 k2 k3 C1 C2 C3
z = sym('z(x)');
p = poly2sym([1 -2 1 0], r); pretty(p)

   3      2
  r  - 2 r  + r
r = solve(p)
%
%
%
```

```
r =

[ 0]
[ 1]
[ 1]

zh = c1 + c2*exp(x) + c3*x*exp(x);
%
L = diff(z,x,3) - 2*diff(z,x,2) + diff(z,x);
pretty(L)

  / 3      \     / 2      \
  |d       |     |d       |   /d      \
  |--- z(x)| - 2 |--- z(x)| + |-- z(x)|
  |  3     |     |  2     |   \dx     /
  \dx      /     \dx      /
zp = x * (a*x + b)  +  x^2 * c*exp(x);
eq0 = subs(L - (x - exp(x)), z, zp);
eq0 = collect(eq0, exp(x));
eq0 = collect(eq0, x)

eq0 =

(-1+2*a)*x+(2*c+1)*exp(x)-4*a+b

[a b c] = solve(-1+2*a, 2*c+1, -4*a+b)

a =

1/2


b =

2


c =

-1/2

%
zp = subs(zp); pretty(zp)

                       2
  x (1/2 x + 2) - 1/2 x  exp(x)
check = subs(L, z, zp)

check =

x-exp(x)

z = zp + zh; z = collect(z, exp(x));
pretty(z)

          2
  (- 1/2 x  + c2 + c3 x) exp(x) + x (1/2 x + 2) + c1
%
sol = dsolve(’D3y - 2*D2y + Dy = x- exp(x)’, ’x’);
sol = collect(sol, exp(x)); pretty(sol)

                               2                2
(C2 + C1(x - 1) + x - 1 - 1/2 x ) exp(x) + 1/2 x  + C3 + 2x
%
% This is equivalent to our answer when we rename constants.
%
sol = subs(sol, [C1 C3], [k3-1, k1]);
sol = simplify(sol); sol = collect(sol, exp(x));
pretty(sol)

                       2                2
(C2 + k3 x - k3 - 1/2 x ) exp(x) + 1/2 x  + k1 + 2 x
sol = subs(sol, C2, k3+k2); pretty(sol)

                  2                2
(k2 + k3 x - 1/2 x ) exp(x) + 1/2 x  + k1 + 2 x
% Voila!
%
% And that retires the side. Good night!
%
echo off; diary off
```