Fall 2003 Math 308/501–502
6 Higher-Order Linear Differential Eqs
6.4/4.6 Variation of Parameters
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Summary

Let $L[y] = y^{(n)}(x) + \sum_{j=1}^{n} p_j(x) y^{(n-j)}(x) = g(x)$ be a nonhomogeneous nth-order linear differential equation in *standard linear form* (SLF). Recall that if y_p is a particular solution of this nonhomogeneous equation and y_h is a general solution to the associated homogeneous equation $L[y] = 0$, then a general solution of the nonhomogeneous equation is given by $y = y_p + y_h$.

Variation of Parameters (VOP)

The following method produces a **general solution** of the nonhomogeneous equation. *First put the differential equation in standard linear form!*

1. Obtain a fundamental set of solutions $\mathbf{y_f} = [y_1, \ldots, y_n]$, a row vector, of $L[y] = 0$. Given a column vector of constants, $\mathbf{c} = [c_1; \ldots; c_n]$, form a general solution of this homogeneous equation, $y_h = \mathbf{y_f c} = \sum_{k=1}^{n} c_k y_k$.
2. Form the column vector $\mathbf{b} = [0; 0; \ldots; 0; g]$, all of whose entries are zeros except the last one, which is the nonhomogeneity g (right-hand side of the nonhomogeneous equation). Compute the Wronskian matrix $\mathbf{M}$ of $\mathbf{y_f}$. Solve $\mathbf{M v_p} = \mathbf{b}$ for $\mathbf{v_p}$ ($\mathbf{v}$ prime), a column vector: **vp = M\b**.
3. Integrate $\mathbf{v_p}$ to obtain $\mathbf{v}$, a column vector. (Don't worry about constants of integration.)
4. A **particular solution** of the nonhomogeneous equation is $y_p = \mathbf{y_f v}$.
5. A **general solution** of the nonhomogeneous equation is $y = y_p + y_h$.

Use MATLAB for your computations!

The operations involved in the variation or parameters procedure are tailor-made for MATLAB: matrix multiplication, solving linear systems, vector integration, etc.

MATLAB Examples

To facilitate computation of the Wronskian matrix, I wrote a function M-file named **wron**. Type "**help wron**" at a MATLAB prompt to learn about it. If you'd like to see the (short) code, type "**type wron**" at a MATLAB prompt.

Example A

Find a general solution of $y'' + 4y = \sec 2t$. (Why can't we use the method of undetermined coefficients here?)

Solution

We paraphrase the steps; computations are done with MATLAB.

1. The characteristic equation, $r^2 + 4 = 0$, has roots $r = \pm 2i$. A fundamental set of solutions is given by the row vector $\mathbf{y_f} = [\cos 2t, \sin 2t]$. A general solution of the homogeneous equation is $y_h = c_1 \cos 2t + c_2 \sin 2t$.
2. Noting that the DE is in SLF, form the column vector $\mathbf{b} = [0; \sec 2t]$ and the Wronskian matrix for $\mathbf{y_f}$: $\mathbf{M} = \begin{bmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{bmatrix}$. Solve the linear system $\mathbf{M v_p} = \mathbf{b}$ to obtain $\mathbf{v_p} = \mathbf{M}\backslash\mathbf{b} = [-\frac{1}{2}\tan 2t; \frac{1}{2}]$.
3. Integrate $\mathbf{v_p}$ to obtain $\mathbf{v} = \int \mathbf{v_p}\, dt = [\frac{1}{4}\ln(\cos 2t); \frac{1}{2}t]$. (Don't worry about constants of integration.)
4. A particular solution of the nonhomogeneous equation is given by the matrix or dot product
$$y_p = \mathbf{y_f v} = \frac{1}{4}\cos 2t \ln(\cos 2t) + \frac{1}{2} t \sin 2t.$$
5. A general solution is $y = y_p + y_h$ or
$$y = \tfrac{1}{4}\cos 2t \ln(\cos 2t) + \tfrac{1}{2} t \sin 2t + c_1 \cos 2t + c_2 \sin 2t.$$

Here is a short MATLAB input file.

```
%
delete s46eA.txt; diary s46eA.txt
clear; clc; close all; echo on
%
% NSS4-4.6/Example A
%
syms c1 c2 r t
p = poly2sym([1 0 4], r); pretty(p)
r = solve(p)
yf = [cos(2*t) sin(2*t)];
c = [c1; c2];
yh = yf*c;
%
M = wron(yf, t)
b = [0; sec(2*t)]
vp = simple(M\b)
v = int(vp, t)
%
yp = yf*v;
y = yp + yh; pretty(y)
%
echo off; diary off
```

Here is the corresponding diary file.

```
%
% NSS4-4.6/Example A
%
syms c1 c2 r t
p = poly2sym([1 0 4], r); pretty(p)

                                     2
                                    r  + 4
r = solve(p)

r =

[  2*i]
[ -2*i]

yf = [cos(2*t) sin(2*t)];
c = [c1; c2];
yh = yf*c;
%
M = wron(yf, t)

M =

[    cos(2*t),    sin(2*t)]
[ -2*sin(2*t),  2*cos(2*t)]

b = [0; sec(2*t)]

b =

[        0]
[ sec(2*t)]

vp = simple(M\b)

vp =

[ -1/2*sin(2*t)/cos(2*t)]
[                    1/2]

v = int(vp, t)

v =

[ 1/4*log(cos(2*t))]
[             1/2*t]

%
yp = yf*v;
y = yp + yh; pretty(y)

    1/4 cos(2 t) log(cos(2 t)) + 1/2 sin(2 t) t
    + cos(2 t) c1 + sin(2 t) c2
%
echo off; diary off
```

Example B

Find a general solution of $x'' - 4x' + 4x = e^{2t}$. (Can we use the method of undetermined coefficients here?)

Solution

The MATLAB input file (here and in every VOP problem) is structurally the same. Here is a diary file.

```
%
% NSS4-4.6/Example B
%
syms c1 c2 r t
p = poly2sym([1 -4 4], r); pretty(p)

                                  2
                                 r  - 4 r + 4
r = solve(p)

r =

[ 2]
[ 2]

yf = [exp(2*t) t*exp(2*t)];
c = [c1; c2];
yh = yf*c;
%
M = wron(yf, t)

M =

[              exp(2*t),            t*exp(2*t)]
[            2*exp(2*t), exp(2*t)+2*t*exp(2*t)]

b = [0; exp(2*t)]

b =

[        0]
[ exp(2*t)]

vp = simple(M\b)

vp =

[ -t]
[  1]

v = int(vp, t)

v =

[ -1/2*t^2]
[        t]

%
yp = yf*v;
y = yp + yh; pretty(y)

                                   2
  1/2 exp(2 t) t  + exp(2 t) c1 + t exp(2 t) c2
%
echo off; diary off
```

Example C

Verify that $y_1(t) = t^{-1}$ and $y_2(t) = t^{-1} \ln t$ are solutions to the homogeneous equation $t^2 y'' + 3ty' + y = 0$, for $t > 0$. Then use variation of parameters to find a general solution to

$$t^2 y'' + 3ty' + y = \frac{1}{t}, \quad \text{for } t > 0$$

Solution

Here is a MATLAB diary file showing that y_1 and y_2 do indeed solve the given homogeneous equation. Also observe that $y_2/y_1 = \ln t$, which is not constant. Thus y_1 and y_2 are linearly independent and hence form a fundamental solution set.

```
%
% NSS4-4.6/Example C1
%
syms t
y = sym('y(t)');
L = t^2*diff(y,t,2) + 3*t*diff(y,t) + y; pretty(L)

                               / 2       \
                            2 |d         |       /d       \
                           t  |--- y(t)| + 3 t |-- y(t)| + y(t)
                              |  2       |       \dt       /
                              \dt        /
check1 = subs(L, y, 1/t)

check1 =

0

check2 = simple(subs(L, y, log(t)/t))

check2 =

0

%
echo off; diary off
```

Pink reminds us that before we get this party started, we MUST ensure that the nonhomogeneous differential equation is in standard linear form!

$$y'' + \frac{3}{t}y' + \frac{1}{t^2}y = \frac{1}{t^3}$$

Accordingly, $g(t) = 1/t^3$, *not* $1/t$. Now do the voodoo that you do so well!

```
%
% NSS4-4.6/Example C2
%
syms c1 c2 r t
% We are GIVEN a fundamental set of solutions!
yf = [1/t log(t)/t];
c = [c1; c2];
yh = yf*c;
%
M = wron(yf, t)

M =

[               1/t,           log(t)/t]
[            -1/t^2, 1/t^2-log(t)/t^2]

b = [0; 1/t^3]

b =

[     0]
[ 1/t^3]

vp = simple(M\b)

vp =

[ -log(t)/t]
[       1/t]

v = int(vp, t)

v =

[ -1/2*log(t)^2]
[        log(t)]

%
yp = yf*v;
y = yp + yh; pretty(y)
%
%
                                  2
                            log(t)      c1    log(t) c2
                        1/2 ------- + ---- + ---------
                               t        t         t
%
echo off; diary off
```

Example D

Find a general solution to the Cauchy-Euler equation

$$L[y] = x^3y''' - 2x^2y'' - 5xy' + 5y = x^{-2} \quad x > 0.$$

(Use the substitution $y = x^r$ to help determine a fundamental solution set for $L[y] = 0$.)

Solution

In Ex. C, the authors handed us a fundamental set of solutions on a plate. Here *we* must do the work! Here's a diary file that shows how we use the suggested substitution to render the needful.

```
%
% NSS4-6.4/Example D1
%
syms c1 c2 c3 r x
y = sym('y(x)');
L = x^3*diff(y,x,3) - 2*x^2*diff(y,x,2) - 5*x*diff(y,x) + 5*y;
pretty(L)

     / 3       \         / 2       \
   3 |d         |      2 |d         |       /d       \
  x  |--- y(x)| - 2 x  |--- y(x)| - 5 x |-- y(x)| + 5 y(x)
     |  3       |        |  2       |       \dx       /
     \dx        /        \dx        /
eq0 = simple(subs(L, y, x^r))

eq0 =

x^r*(r^3-5*r^2-r+5)

r = solve(r^3-5*r^2-r+5)

r =

[ -1]
[  1]
[  5]

yf = [1/x, x, x^5];
%
echo off; diary off
```

Now we're all set to go. Finish the problem.

```
%
% NSS4-6.4/Example D2
%
syms c1 c2 c3 x
% Here's our fundamental set of solutions
% that we determined earlier.
yf = [1/x, x, x^5];
c = [c1; c2; c3]; yh = yf*c;
%
M = wron(yf, x)
```

```
M =

[    1/x,       x,    x^5]
[ -1/x^2,       1, 5*x^4]
[  2/x^3,       0, 20*x^3]

b = [0; 0; 1/x^5] % Put the DE in SLF! Don't forget!

b =

[     0]
[     0]
[ 1/x^5]

% # elements in column vector b equals order of DE: 3.
vp = simple(M\b)

vp =

[ 1/12/x^2]
[ -1/8/x^4]
[ 1/24/x^8]

v = int(vp, x)

v =

[    -1/12/x]
[   1/24/x^3]
[ -1/168/x^7]

%
yp = yf*v;
y = yp + yh; pretty(y)

                                1      c1             5
                      - 1/21 ---- + ---- + x c2 + x  c3
                                 2     x
                                x
%
echo off; diary off
```

Example E

Find the unique solution of the differential equation in Example D that satisfies these ICs: $y(1) = 1,\ y'(1) = -2,\ y''(1) = 6.$

Solution

Embrace the matrix-vector way! (Or suffer. . .) Questions? Ask!

```
%
% NSS4-6.4/Example E
%
syms c1 c2 c3 x
y = sym('y(x)');
% Here is our fundamental set yf of solutions from Ex. D,
% together with the particular solution yp of the
% nonhomogeneous equation.
yf = [1/x, x, x^5]; pretty(yf)

                               [              5]
                               [1/x    x    x ]
yp = -1 / (21*x^2); pretty(yp)

                                          1
                                 - 1/21 ----
                                          2
                                         x
%
v = [yf yp] % Push WRON beyond its design specs...

v =

[       1/x,          x,       x^5, -1/21/x^2]

M = wron(v, x); M = subs(M, x, sym(1))

M =

[     1,     1,     1, -1/21]
[    -1,     1,     5,  2/21]
[     2,     0,    20,  -2/7]
[    -6,     0,    60,   8/7]

a = M(1:3, 4)   % Extract

a =

[ -1/21]
[  2/21]
[  -2/7]

M = M(1:3, 1:3) % the needful.

M =

[  1,  1,  1]
[ -1,  1,  5]
[  2,  0, 20]

b = [1; -2; 6]  % <- our ICs
b =
     1
    -2
     6
%
c = M\(b-a) % BANG!

c =

[   11/6]
[ -11/12]
[  11/84]

y = yp + yf*c; pretty(y)

                          1                 11      11  5
                 - 1/21 ---- + 11/6 1/x - -- x + -- x
                          2                 12      84
                         x
%
sol = dsolve('x^3*D3y - 2*x^2*D2y - 5*x*Dy + 5*y = 1/x^2', ...
    'y(1)=1', 'Dy(1)=-2', 'D2y(1)=6', 'x');
pretty(expand(sol)) % And that retires the side!

                          1                 11      11  5
                 - 1/21 ---- + 11/6 1/x - -- x + -- x
                          2                 12      84
                         x
%
echo off; diary off
```