Fall 2003 Math 308/501–502
7 Laplace Transforms
7.2 Definition of the Laplace Transform
Wed, 08/Oct ©2003, Art Belmonte

Summary

Definitions

- Given a function f of t, the **Laplace transform** of f is a function $\mathcal{L}\{f\}(s) = F(s)$ defined by

  $$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{T\to\infty}\int_0^T f(t)e^{-st}\,dt,$$

  when this limit exists. Here $s > 0$ and may be further restricted depending on f. The transform is an **operator** which acts on a function to produce yet another function.

- A function f is **piecewise continuous** on $(0, \infty)$ if it has only finitely many jump discontinuities on any finite subinterval of $(0, \infty)$.

- In a similar manner, f is **piecewise differentiable** on $(0, \infty)$ if it is continuous and its derivative is piecewise continuous.

- A function f is of **exponential order** if $|f(t)| \le Ce^{at}$ for $t > 0$, where C and a are constants.

Existence Theorem for Laplace Transforms

If f is a function defined on $[0, \infty)$ that is piecewise continuous and of exponential order (say $|f(t)| \le Ce^{at}$), then the Laplace transform $\mathcal{L}\{f\}(s)$ exists for $s > a$ (at least).

Linearity of the Laplace Transform

Let the Laplace transforms of f, f_1, and f_2 exist for $s > \alpha$ and let c be any constant. Then we have

$$\begin{aligned} \mathcal{L}\{f_1 + f_2\} &= \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\} \\ \mathcal{L}\{cf\} &= c\mathcal{L}\{f\} \end{aligned}$$

Notes

When computing Laplace transforms strictly with a pencil, we use integration by parts, L'Hospital's rule, etc. On the other hand, the MATLAB Symbolic Math Toolbox (SMT) command **laplace** computes Laplace transforms at one fell swoop. In between these two extremes is a middle ground, where one mimicks hand work semiautomatically with the SMT commands **int**, **subs**, and **limit**. This is how ***Hand Examples*** below were computed. They were verified via **laplace**.

Small Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$	Restrictions
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin bt$	$\frac{b}{s^2+b^2}$	$s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}$	$s > 0$
$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$

Hand Examples

352/Example 1

Use the definition to compute the Laplace transform of 1.

Solution

Compute the integral involved over a finite interval.

$$\begin{aligned} \int_0^T 1\cdot e^{-st}\,dt &= \left.\left(-\frac{e^{-st}}{s}\right)\right|_0^T \\ &= -\frac{e^{-sT}}{s} - \left(-\frac{1}{s}\right) \end{aligned}$$

Let $T \to \infty$ to obtain $\int_0^\infty 1\cdot e^{-st}\,dt$. For $s > 0$, we have

$$\lim_{T\to\infty}\left(\frac{1}{s} - \frac{e^{-sT}}{s}\right) = \frac{1}{s}$$

Therefore, $\mathcal{L}\{1\}(s) = \frac{1}{s}$.

Example B

Use the definition to compute the Laplace transform of $\sin 3t$.

Solution

Compute the integral involved over a finite interval.

$$\begin{aligned} \int_0^T \sin 3t\cdot e^{-st}\,dt &= \left.\left(-\frac{e^{-st}(s\sin 3t + 3\cos 3t)}{s^2+9}\right)\right|_0^T \\ &= \frac{3 - se^{-sT}\sin 3T - 3e^{-sT}\cos 3T}{s^2+9} \end{aligned}$$

Let $T \to \infty$ to obtain $\int_0^\infty \sin 3t \cdot e^{-st}\,dt$. For $s > 0$, we have

$$\lim_{T\to\infty} \frac{3 - se^{-sT}\sin 3T - 3e^{-sT}\cos 3T}{s^2+9} = \frac{3}{s^2+9}$$

Therefore, $\mathcal{L}\{\sin 3t\}(s) = \dfrac{3}{s^2+9}$.

Example C

Use the definition to compute the Laplace transform of te^{-3t}.

Solution

Compute the integral involved over a finite interval.

$$\begin{aligned} \int_0^T te^{-3t} \cdot e^{-st}\,dt &= \left. \left(-\frac{e^{-(s+3)t}(st+3t+1)}{(s+3)^2} \right) \right|_0^T \\ &= \frac{1 - e^{-(s+3)T}(sT+3T+1)}{(s+3)^2} \end{aligned}$$

Let $T \to \infty$ to obtain $\int_0^\infty te^{-3t} \cdot e^{-st}\,dt$. For $s > 0$, we have

$$\lim_{T\to\infty} \frac{1 - e^{-(s+3)T}(sT+3T+1)}{(s+3)^2} = \frac{1}{(s+3)^2}$$

Therefore, $\mathcal{L}\left\{te^{-3t}\right\}(s) = \dfrac{1}{(s+3)^2}$.

Example D

Use the definition to compute the Laplace transform of $\cos bt$.

Solution

Compute the integral involved over a finite interval.

$$\begin{aligned} \int_0^T \cos bt \cdot e^{-st}\,dt &= \left. \left(\frac{e^{-st}(b\sin bt - s\cos bt)}{s^2+\omega^2} \right) \right|_0^T \\ &= \frac{s + e^{-sT}(b\sin bT - s\cos bT)}{s^2+b^2} \end{aligned}$$

Let $T \to \infty$ to obtain $\int_0^\infty \cos bt \cdot e^{-st}\,dt$. For $s > 0$, we have

$$\lim_{T\to\infty} \frac{s + e^{-sT}(b\sin bT - s\cos bT)}{s^2+b^2} = \frac{s}{s^2+b^2}$$

Therefore, $\mathcal{L}\{\cos bt\}(s) = \dfrac{s}{s^2+b^2}$.

359/16

Compute the Laplace transform of $t^2 - 3t - 2e^{-t}\sin 3t$ via table lookup and linearity.

Solution

For $s > 0$, we have

$$\begin{aligned} \mathcal{L}\left\{t^2 - 3t - 2e^{-t}\sin 3t\right\} &= \mathcal{L}\left\{t^2\right\} - 3\mathcal{L}\{t\} - 2\mathcal{L}\left\{e^{-t}\sin 3t\right\} \\ &= \frac{2!}{s^{2+1}} - 3\frac{1!}{s^{1+1}} - 2\frac{3}{(s+1)^2+3^2} \\ &= \frac{2}{s^3} - \frac{3}{s^2} - \frac{6}{(s+1)^2+9} \end{aligned}$$

Example F

Compute the Laplace transform of $\cos^2 2t + \sin 5t$.

Solution

For $s > 0$, we have

$$\begin{aligned} \mathcal{L}\left\{\cos^2 2t + \sin 5t\right\} &= \mathcal{L}\left\{\tfrac{1}{2}(1+\cos 4t) + \sin 5t\right\} \\ &= \tfrac{1}{2}(\mathcal{L}\{1\} + \mathcal{L}\{\cos 4t\}) + \mathcal{L}\{\sin 5t\} \\ &= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4^2}\right) + \frac{5}{s^2+5^2} \\ &= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right) + \frac{5}{s^2+25} \end{aligned}$$

Example G

Show that $f(t) = \sin\left(t^2\right) + t^4e^{6t}$ is of exponential order.

Solution

Let $\alpha = 7$, $M = 25$, and $T = 1$. Then for all $t \geq T = 1$, we have

$$\begin{aligned} |f(t)| &= \left|\sin\left(t^2\right) + t^4e^{6t}\right| \\ &\leq \left|\sin\left(t^2\right)\right| + t^4e^{6t} \\ &\leq 1 + 24\left(1 + t + \tfrac{t^2}{2!} + \tfrac{t^3}{3!} + \tfrac{t^4}{4!} + \cdots\right)e^{6t} \\ &\leq e^{7t} + 24e^te^{6t} \\ &= 25e^{7t} = Me^{\alpha t}. \end{aligned}$$

Therefore, $f(t)$ is of exponential order.

MATLAB Examples

352/Example 1 [revisited]

Compute the Laplace transform of 1 semiautomatically.

Solution

Here is a diary file showing the needful. We'll type input commands into the MATLAB editor together in class and run the code. To do another problem, all you need to do is change the definition of f.

```
%
% NSS4-352/Example 1
%
syms T t; syms s positive
f = 1;
I = int(f*exp(-s*t), t); pretty(I)

                                  exp(-s~ t)
                                - ----------
                                      s~
v = subs(I,t,T) - subs(I,t,0);
pretty(v)

                               exp(-s~ T)    1
                             - ---------- + ----
                                   s~        s~
F = limit(v, T, inf); pretty(F)

                                     1
                                    ----
                                     s~
syms s unreal
% Remove assumption(s) on s.
% Check
% No joy in Mudville...
% F = laplace(f); pretty(F) % NO t in f; so stick one in!
F = laplace(f + 0*t); pretty(F)

                                     1/s
%

echo off; diary off
```

Example B [revisited]

Compute the Laplace transform of $\sin 3t$ semiautomatically.

Solution

Reload and squeeze the trigger. (Just redefine f.)

```
%
% NSS4-7.2/Example B
%
syms T t; syms s positive
f = sin(3*t);
I = int(f*exp(-s*t), t); pretty(I)

     exp(-s~ t) cos(3 t)   s~ exp(-s~ t) sin(3 t)
  -3 ------------------- - ---------------------
             2                     2
           s~  + 9               s~  + 9
v = subs(I,t,T) - subs(I,t,0);
pretty(v)

     exp(-s~ T) cos(3 T)   s~ exp(-s~ T) sin(3 T)      3
  -3 ------------------- - --------------------- + -------
             2                     2                  2
           s~  + 9               s~  + 9            s~  + 9
%
%
%
F = limit(v, T, inf); pretty(F)

                                      3
                                   -------
                                      2
                                    s~  + 9
syms s unreal
% Remove assumption(s) on s.
% Check
F = laplace(f); pretty(F)

                                      3
                                    ------
                                     2
                                    s  + 9
%

echo off; diary off
```

Example C [revisited]

Compute the Laplace transform of te^{-3t} semiautomatically.

Solution

Round up the Usual Suspects. . .

```
%
% NSS4-7.2/Example C
%
syms T t; syms s positive
f = t*exp(-3*t);
I = int(f*exp(-s*t), t); pretty(I)

  (-3 - s~) t exp((-3 - s~) t) - exp((-3 - s~) t)
  -----------------------------------------------
                                 2
                    (-3 - s~)
v = subs(I,t,T) - subs(I,t,0);
pretty(v)

(-3 - s~) T exp((-3 - s~) T) - exp((-3 - s~) T)       1
----------------------------------------------- + ----------
                               2                           2
                  (-3 - s~)                       (-3 - s~)
F = limit(v, T, inf); pretty(F)
%
%

                                 1
                           --------------
                                        2
                           9 + 6 s~ + s~
syms s unreal
% Remove assumption(s) on s.
% Check
F = laplace(f); pretty(F)

                                 1
                              --------
                                     2
                              (s + 3)
%

echo off; diary off
```

Example D [revisited]

Compute the Laplace transform of $\cos bt$ semiautomatically.

Solution

By now you know the drill. . .

```
%
% NSS4-7.2/Example D
%
syms b T t; syms s positive
f = cos(b*t);
I = int(f*exp(-s*t), t); pretty(I)

    s~ exp(-s~ t) cos(b t)   b exp(-s~ t) sin(b t)
  - --------------------- + ---------------------
            2    2                  2    2
          s~  + b                 s~  + b
v = subs(I,t,T) - subs(I,t,0);
pretty(v)

  s~ exp(-s~ T) cos(b T)   b exp(-s~ T) sin(b T)      s~
```

```
- --------------------- + -------------------- + --------
         2    2                  2    2                2    2
       s~  + b                 s~  + b               s~  + b
F = limit(v, T, inf); pretty(F)

                                   s~
                                --------
                                  2    2
                                s~  + b
syms s unreal
% Remove assumption(s) on s.
% Check
F = laplace(f); pretty(F)

                                    s
                                 -------
                                  2    2
                                 s  + b
%
echo off; diary off
```

Example B [Laplace transform via a table or MATLAB]

Compute the Laplace transform of $\sin 3t$ via table lookup and/or MATLAB. This is Example B revisited one last time.

Solution

On page 358 (or from the inside back cover), we have $\mathcal{L}\{\sin bt\} = \dfrac{b}{s^2+b^2}$ for $s > 0$. So $\mathcal{L}\{\sin 3t\} = \dfrac{3}{s^2+9}$.
(The MATLAB check via **laplace** was done during our last visit!)

Example H

Find the Laplace transform of the piecewise function

$$f(t) = \begin{cases} t, & 0 \le t < 3 \\ 3, & t \ge 3 \end{cases}$$

Solution

We split the integral into two pieces and proceed by brute force. (We'll have a more elegant way of dispatching Laplace transforms of piecewise functions in Section 7.6.)

$$\int_0^\infty f(t)e^{-st}\,dt = \int_0^3 te^{-st}\,dt + \int_3^\infty 3e^{-st}\,dt$$

```
%
% NSS4-7.2/Example E
%
syms T t; syms s positive
f1 = t; f2 = 3;
I1 = int(f1*exp(-s*t), t, 0, 3); pretty(I1)

                       exp(-3 s~) + 3 exp(-3 s~) s~ - 1
                     - --------------------------------
                                       2
                                     s~
I2 = int(f2*exp(-s*t), t, 3, T); pretty(I2)

                            exp(-s~ T) - exp(-3 s~)
                         -3 -----------------------
                                      s~
v = I1 + I2; pretty(v)

         exp(-3 s~) + 3 exp(-3 s~) s~ - 1
       - --------------------------------
                         2
                       s~

             exp(-s~ T) - exp(-3 s~)
         - 3 -----------------------
                       s~
F = limit(v, T, inf); pretty(F)

                                         3
                             -1 + exp(s~)
                             -------------
                               2        3
                             s~  exp(s~)
F = simple(F); pretty(F)

                                 1          1
                           - ------------ + ---
                               2        3     2
                             s~  exp(s~)    s~
equiv = (1 - exp(-3*s)) / s^2; pretty(equiv)

                            1 - exp(-3 s~)
                            --------------
                                   2
                                 s~
syms s unreal % Remove assumption(s) on s.
%
echo off; diary off
```