Fall 2003 Math 308/501–502
7 Laplace Transforms
7.3 Properties of Laplace Transforms
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Summary

Looking up Laplace transforms in tables is aided by employing various properties that the Laplace transform satisfies. This also aids a computer when calculating transforms!

Linearity Property

For constants α and β and piecewise continuous functions f and g of exponential order, we have $\mathcal{L}\{\alpha f + \beta g\} = \alpha\mathcal{L}\{f\} + \beta\mathcal{L}\{g\}$.

Laplace transforms of derivatives

Let $y^{(k)}$, $k = 0, 1, \ldots n-1$, be piecewise differentiable and continuous functions of exponential order. Let $y^{(n)}$ be piecewise continuous and of exponential order. If $\mathcal{L}\{y(t)\}(s) = Y(s)$, then

$$\mathcal{L}\left\{y^{(n)}\right\}(s) = s^n Y(s) - \sum_{k=1}^{n} s^{n-k} y^{(k-1)}(0)$$

In particular, for $n = 1$, we have

$$\mathcal{L}\left\{y'\right\}(s) = sY(s) - y(0),$$

and for $n = 2$, this yields

$$\mathcal{L}\left\{y''\right\}(s) = s^2 Y(s) - sy(0) - y'(0).$$

First translation or shifting property

Let f be a piecewise continuous function of exponential order with transform $\mathcal{L}\{f(t)\}(s) = F(s)$ and let c be a constant. Then

$$\mathcal{L}\left\{e^{ct} f(t)\right\}(s) = F(s-c) \text{ for } s > c$$

Second translation or shifting property

If $\mathcal{L}\{f\}(s) = F(s)$ and $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$, then

$$\mathcal{L}\{g(t)\}(s) = G(s) = e^{-as}F(s)$$

Change of scale property

If $\mathcal{L}\{f\}(s) = F(s)$, then

$$\mathcal{L}\{f(at)\}(s) = \frac{1}{a}F\left(\frac{s}{a}\right)$$

Relation between Laplace transform and its derivatives

Let f be a piecewise continuous function of exponential order with transform $\mathcal{L}\{f(t)\}(s) = F(s)$. Then

$$\mathcal{L}\left\{t^n f(t)\right\}(s) = (-1)^n F^{(n)}(s)$$

In particular, for $n = 1$ we have

$$\mathcal{L}\{tf(t)\}(s) = -F'(s)$$

Hand Examples

Example A

Compute the Laplace transform of t^n for nonnegative integers n.

Solution

From the first example in the Section 7.2 lecture handout, we have $\mathcal{L}\left\{t^0\right\} = \mathcal{L}\{1\} = \int_0^\infty 1 \cdot e^{-st}\,dt = \dfrac{1}{s}$. Thus

$$\begin{aligned}
\mathcal{L}\left\{t^1\right\} &= -\frac{d}{ds}(s^{-1}) = s^{-2} = \frac{1}{s^2}, \\
\mathcal{L}\left\{t^2\right\} &= -\frac{d}{ds}(s^{-2}) = 2s^{-3} = \frac{2}{s^3}, \\
\mathcal{L}\left\{t^3\right\} &= -\frac{d}{ds}(2s^{-3}) = 6s^{-4} = \frac{6}{s^4},
\end{aligned}$$

and by induction $\mathcal{L}\left\{t^n\right\} = \dfrac{n!}{s^{n+1}}$. All are defined for $s > 0$.

Example B

Compute the Laplace transform of $y(t) = 3 - 5t - 11t^3$.

Solution

We have $\mathcal{L}\left\{3 - 5t - 11t^3\right\} = 3\mathcal{L}\{1\} - 5\mathcal{L}\{t\} - 11\mathcal{L}\left\{t^3\right\}$
$= 3\left(\dfrac{1}{s}\right) - 5\left(\dfrac{1}{s^2}\right) - 11\left(\dfrac{6}{s^4}\right) = \dfrac{3}{s} - \dfrac{5}{s^2} - \dfrac{66}{s^4}.$

Example C

Compute the Laplace transform of $y(t) = 2\sin 3t + 3\cos 5t$.

Solution

Recall from the table on the inside back cover that $\mathcal{L}\{\sin bt\} = \dfrac{b}{s^2+b^2}$ and $\mathcal{L}\{\cos bt\} = \dfrac{s}{s^2+b^2}$. Hence $\mathcal{L}\{2\sin 3t + 3\cos 5t\} = 2\mathcal{L}\{\sin 3t\} + 3\mathcal{L}\{\cos 5t\}$ $= 2\,\dfrac{3}{s^2+3^2} + 3\,\dfrac{s}{s^2+5^2} = \dfrac{6}{s^2+9} + \dfrac{3s}{s^2+25}$.

Example D

For the function $y(t) = e^{2t}$, verify that $\mathcal{L}\left\{y'\right\}(s) = sY(s) - y(0)$. Here $\mathcal{L}\{y\}(s) = Y(s)$. Use properties and your table.

Solution

$Y(s) = \mathcal{L}\left\{e^{2t}\right\} = \dfrac{1}{s-2}$ and $\mathcal{L}\left\{y'\right\}(s) = \mathcal{L}\left\{2e^{2t}\right\} = \dfrac{2}{s-2}$.
Thus $sY(s) - y(0) = \dfrac{s}{s-2} - 1 = \dfrac{2}{s-2} = \mathcal{L}\left\{y'\right\}(s)$.

Example E

For the function $y(t) = \sin 2t$, verify that

$$\mathcal{L}\left\{y''\right\}(s) = s^2Y(s) - sy(0) - y'(0)$$

Here $\mathcal{L}\{y\}(s) = Y(s)$. Use properties and your table.

Solution

Now $Y(s) = \mathcal{L}\{\sin 2t\} = \dfrac{2}{s^2+4}$ and
$\mathcal{L}\left\{y''\right\} = \mathcal{L}\{-4\sin 2t\} = \dfrac{-8}{s^2+4}$. Thus
$s^2Y(s) - sy(0) - y'(0) = \dfrac{2s^2}{s^2+4} - 0 - 2 = \dfrac{-8}{s^2+4} = \mathcal{L}\left\{y''\right\}(s)$.

Example F

Use properties of the Laplace transform to change the IVP $y' + 5y = t^2 + 2t + 3$, $y(0) = 0$, into an *algebraic* equation involving $Y(s) = \mathcal{L}\{y(t)\}$. Then solve for $Y(s)$.

Solution

1. Take the Laplace transform of each side of the DE.

$$\begin{aligned} y' + 5y &= t^2 + 2t + 3 \\ sY(s) - y(0) + 5Y(s) &= \frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s} \end{aligned}$$

2. Substitute for the initial condition(s).

$$sY(s) + 5Y(s) = \frac{2 + 2s + 3s^2}{s^3}$$

3. Solve for $Y(s)$, the Laplace transform of y.

$$Y(s) = \frac{3s^2 + 2s + 2}{s^3(s+5)}$$

Example G

Use properties of the Laplace transform to change the IVP $y'' + y' + 2y = \cos 2t + \sin 3t$, $y(0) = -1$, $y'(0) = 1$, into an *algebraic* equation involving $Y(s) = \mathcal{L}\{y(t)\}$. Solve for $Y(s)$.

Solution

1. Take the Laplace transform of $y'' + y' + 2y = \cos 2t + \sin 3t$.

$$s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) + 2Y(s) = \frac{s}{s^2+4} + \frac{3}{s^2+9}$$

2. Substitute for the initial conditions.

$$\begin{aligned} s^2Y(s) + s - 1 + sY(s) + 1 + 2Y(s) &= \frac{s(s^2+9) + 3(s^2+4)}{(s^2+4)(s^2+9)} \\ (s^2 + s + 2)Y(s) + s &= \frac{s(s^2+9) + 3(s^2+4)}{(s^2+4)(s^2+9)} \end{aligned}$$

3. Solve for $Y(s)$.

$$Y(s) = \frac{s(s^2+9) + 3(s^2+4)}{(s^2+4)(s^2+9)(s^2+s+2)} - \frac{s}{s^2+s+2}$$

MATLAB Examples

Example B [revisited]

Use MATLAB to compute the Laplace transform of $y(t) = 3 - 5t - 11t^3$.

Solution

It's easy: just use **laplace**! You may verify other Laplace transforms the same way.

```
% NSS4-7.3/Example B
%
syms t
y = 3 - 5*t - 11*t^3;
Y = laplace(y); pretty(Y)

        5      66
  3/s - ---- - ----
         2      4
        s      s
```

Example H

Compute the Laplace transform of $e^{-2t}(2t+3)$ with MATLAB.

Solution

Apply **laplace**, then simplify.

```
%
% NSS4-7.3/Example H
%
syms t
y = exp(-2*t) * (2*t+3);
Y = laplace(y); pretty(Y)

     2               1
  -------- + 3/2 ---------
        2        1 + 1/2 s
  (s + 2)
Y = simple(Y); pretty(Y)

  8 + 3 s
  --------
        2

  (s + 2)
```

Example I

Compute the Laplace transform of $t\sin 3t$ with MATLAB.

Solution

Here is a diary file.

```
%
% NSS4-7.3/Example I
%
syms t
y = t*sin(3*t);
Y = laplace(y); pretty(Y)

        s
  6 ---------
       2    2
    (s  + 9)
%
echo off; diary off
```

Example J

Use MATLAB to change the initial value problem $y'' + 2y' + 5y = t^2 e^{-t}$, $y(0) = 1$, $y'(0) = -2$, into an algebraic equation involving $Y(s) = \mathcal{L}\{y(t)\}$. Then solve for $Y(s)$.

Solution

Study this example carefully! It gives a semiautomatic way of doing what we did in by hand in Examples F and G. Here is the script M-file.

```
%
% NSS4-7.3/Example J
%
syms s t Ys
y = sym(’y(t)’);
de0 = diff(y,t,2) + 2*diff(y,t) + 5*y ...
    - tˆ2*exp(-t); pretty(de0)
ltde = laplace(de0);  % Suppress DE output,
ltde = chiclet(ltde) % then "filter" it.
pretty(ltde)
eq0 = subs(ltde, {’y(0)’ ’Dy(0)’}, {1 -2}); pretty(eq0)
Ys = solve(eq0, Ys); pretty(Ys)
%
echo off; diary off
```

Here is the diary file produced by the preceding script M-file.

```
%
% NSS4-7.3/Example J
%
syms s t Ys
y = sym(’y(t)’);
de0 = diff(y,t,2) + 2*diff(y,t) + 5*y ...
    - tˆ2*exp(-t); pretty(de0)

                / 2       \
                |d        |     /d      \           2
                |--- y(t)| + 2 |-- y(t)| + 5 y(t) - t  exp(-t)
                |  2      |     \dt     /
                \dt       /
ltde = laplace(de0);  % Suppress DE output,
ltde = chiclet(ltde) % then "filter" it.

ltde =

s*(s*Ys-y(0))-Dy(0)+2*s*Ys-2*y(0)+5*Ys-2/(s+1)ˆ3

pretty(ltde)

                                                             2
  s (s Ys - y(0)) - Dy(0) + 2 s Ys - 2 y(0) + 5 Ys - --------
                                                            3
                                                     (s + 1)
eq0 = subs(ltde, {’y(0)’ ’Dy(0)’}, {1 -2}); pretty(eq0)

                                                   2
                  s (s Ys - 1) + 2 s Ys + 5 Ys - --------
                                                        3
                                                 (s + 1)
Ys = solve(eq0, Ys); pretty(Ys)

                             4      3      2
                            s  + 3 s  + 3 s  + s + 2
                            ------------------------
                                   3   2
                            (s + 1)  (s  + 2 s + 5)
%
echo off; diary off
```

Example F [revisited]

```
%
% NSS4-7.3/Example F
%
syms s t Ys
y = sym(’y(t)’);
de0 = diff(y,t) + 5*y ...
    - ( tˆ2 + 2*t + 3 ); pretty(de0)

                       /d      \              2
                       |-- y(t)| + 5 y(t) - t  - 2 t - 3
                       \dt     /
ltde = laplace(de0);
ltde = chiclet(ltde);
pretty(ltde)
```

```
                                            2      2
                    s Ys - y(0) + 5 Ys - ---- - ---- - 3/s
                                            3      2
                                           s      s
eq0 = subs(ltde, 'y(0)', 0); pretty(eq0)

                                         2      2
                    s Ys + 5 Ys - ---- - ---- - 3/s
                                         3      2
                                        s      s
Ys = solve(eq0, Ys); pretty(Ys)

                                    2
                                 3 s  + 2 + 2 s
                                 --------------
                                    3
                                   s  (5 + s)
%
echo off; diary off
```

Example G [revisited]

```
%
% NSS4-7.3/Example G
%
syms s t Ys
y = sym('y(t)');
de0 = diff(y,t,2) + diff(y,t) + 2*y ...
    - ( cos(2*t) + sin(3*t) ); pretty(de0)

  / 2        \
  |d         |   /d       \
  |--- y(t)| + |-- y(t)| + 2 y(t) - cos(2 t) - sin(3 t)
  |  2       |   \dt      /
  \dt        /
ltde = laplace(de0);
ltde = chiclet(ltde);
pretty(ltde)


  s (s Ys - y(0)) - Dy(0) + s Ys - y(0) + 2 Ys

      s        3
  - ------ - ------
     2        2
    s  + 4   s  + 9
eq0 = subs(ltde, {'y(0)' 'Dy(0)'}, {-1 1}); pretty(eq0)

                                       s        3
  s (s Ys + 1) + s Ys + 2 Ys - ------ - ------
                                      2        2
                                     s  + 4   s  + 9
Ys = solve(eq0, Ys); pretty(Ys)

                           5       3              2
                          s  + 12 s  + 27 s - 3 s  - 12
                        - -----------------------------
                            2        2                 2
                          (s  + 4) (s  + 9) (s + 2 + s )
%
echo off; diary off
```