Fall 2003 Math 308/501–502
7 Laplace Transforms
7.4 Inverse Laplace Transform
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Summary

Theorem

If f and g are continuous functions whose Laplace transforms are such that $\mathcal{L}\{f\}(s) = \mathcal{L}\{g\}(s)$ for $s > a$, then $f(t) = g(t)$ for all $t > 0$.

Definition

For a continuous function f of exponential order whose Laplace transform is F, we call f the **inverse Laplace transform** of F and write $f = \mathcal{L}^{-1}\{F\}$. (The preceding theorem is what makes this definition possible.)

Linearity Property

For constants α and β and inverse Laplace transforms $\mathcal{L}^{-1}\{F\} = f$ and $\mathcal{L}^{-1}\{G\} = g$, we have

$$\mathcal{L}^{-1}\{\alpha F + \beta G\} = \alpha\mathcal{L}^{-1}\{F\} + \beta\mathcal{L}^{-1}\{G\} = \alpha f + \beta g$$

Short table of [inverse] Laplace transforms

#	$f(t) = \mathcal{L}^{-1}\{F\}(t)$	$\mathcal{L}\{f\}(s) = F(s)$
1.	1	$\frac{1}{s}, \quad s > 0$
2.	t^n	$\frac{n!}{s^{n+1}}, \quad s > 0$
3.	$\sin at$	$\frac{a}{s^2 + a^2}, \quad s > 0$
4.	$\cos at$	$\frac{s}{s^2 + a^2}, \quad s > 0$
5.	e^{at}	$\frac{1}{s - a}, \quad s > a$
6.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2 + b^2}, \quad s > a$
7.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2 + b^2}, \quad s > a$
8.	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$

Notes

When computing inverse Laplace transforms strictly with a pencil, we typically use partial fraction decomposition, completing the square, and algebraic manipulation before finally employing table lookup. On the other hand, the MATLAB Symbolic Math Toolbox (SMT) command **ilaplace** computes inverse Laplace transforms at one fell swoop. In between these two extremes is a middle ground, where one mimicks hand work semiautomatically with the new **cpf** (convert to partial fractions) MATLAB rountine I wrote for you along the SMT command **expand**. This is how some of the ***Hand Examples*** below were computed, as shown in MATLAB files in the ***MATLAB Examples***. Naturally the inverse Laplace transforms were verified via **ilaplace**.

The new **cpf** routine is automatically available to you on your CalcLab account. It also works with the version of MATLAB in TAMU open access labs as well as the Student Version at home.

Hand Examples

Example A

Compute the inverse Laplace transform of $Y(s) = \dfrac{5s}{s^2+9}$.

Solution

We have
$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\dfrac{5s}{s^2+9}\right\} = 5\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+3^2}\right\} = 5\cos 3t.$

Example B

Find the inverse Laplace transform of $Y(s) = \dfrac{1}{s^5} - \dfrac{5-9s}{s^2+100}$.

Solution

Now $Y(s) = \dfrac{1}{4!}\,\dfrac{4!}{s^{4+1}} - \dfrac{1}{2}\,\dfrac{10}{s^2+10^2} + 9\,\dfrac{s}{s^2+10^2}$. Therefore,
$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{24}\mathcal{L}^{-1}\left\{\frac{4!}{s^{4+1}}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{10}{s^2+10^2}\right\} + 9\mathcal{L}^{-1}\left\{\frac{s}{s^2+10^2}\right\}$
$= \frac{1}{24}t^4 - \frac{1}{2}\sin 10t + 9\cos 10t.$

Example C

Compute the inverse Laplace transform of $Y(s) = \dfrac{4s}{(s-1)^2+4}$.

Solution

Now $Y(s) = \dfrac{4(s-1)+4}{(s-1)^2+4} = 4\,\dfrac{(s-1)}{(s-1)^2+2^2} + 2\,\dfrac{2}{(s-1)^2+2^2}$.
Hence
$\mathcal{L}^{-1}\{Y(s)\} = 4\mathcal{L}^{-1}\left\{\dfrac{(s-1)}{(s-1)^2+2^2}\right\} + 2\mathcal{L}^{-1}\left\{\dfrac{2}{(s-1)^2+2^2}\right\}$
$= 4e^t\cos 2t + 2e^t\sin 2t.$

Example D

Find the inverse Laplace transform of $Y(s) = \dfrac{5-2s}{s^2-2s+5}$.

Solution

We have
$Y(s) = \dfrac{-2(s-1)+3}{(s-1)^2+2^2} = -2\,\dfrac{(s-1)}{(s-1)^2+2^2} + \dfrac{3}{2}\,\dfrac{2}{(s-1)^2+2^2}$.
Thus
$\mathcal{L}^{-1}\{Y(s)\} = -2\mathcal{L}^{-1}\left\{\dfrac{(s-1)}{(s-1)^2+2^2}\right\} + \dfrac{3}{2}\mathcal{L}^{-1}\left\{\dfrac{2}{(s-1)^2+2^2}\right\}$
$= -2e^t\cos 2t + \frac{3}{2}e^t\sin 2t$.

Example E

Calculate the inverse Laplace transform of $Y(s) = \dfrac{4s+15}{2s^2+3s}$.

Solution

First split $Y(s)$ into a sum of partial fractions.

$$\begin{aligned}\frac{4s+15}{s(2s+3)} &= \frac{a}{s} + \frac{b}{2s+3}\\ 4s+15 &= a(2s+3)+bs\\ 4s+15 &= (2a+b)s+3a\end{aligned}$$

So $2a+b=4$ and $3a=15$. Thus $a=5$ and $b=4-2a=-6$.
Therefore $Y(s) = \dfrac{5}{s} - \dfrac{6}{2s+3} = 5\,\dfrac{1}{s} - 3\,\dfrac{1}{s-(-\frac{3}{2})}$. Hence
$\mathcal{L}^{-1}\{Y(s)\} = 5\mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\} - 3\mathcal{L}^{-1}\left\{\dfrac{1}{s-(-\frac{3}{2})}\right\} = 5-3e^{-\frac{3}{2}t}$.

MATLAB Examples

You may check any inverse Laplace transform directly with **ilaplace**, with or without first computing a partial fraction decomposition.

Example E [revisited]

Calculate the inverse Laplace transform of $Y(s) = \dfrac{4s+15}{2s^2+3s}$.

Solution

Here we use **cpf** to help with partial fractions. Then we take the result (and the original form of $Y(s)$) and compute the inverse Laplace transform. Either way, we win!

```
%
% NSS4-7.4/ Example E
%
syms a b s
q = (4*s+15) / (2*s^2+3*s); pretty(q)

                                          4 s + 15
                                         ----------
                                             2
                                         2 s  + 3 s
F = a/s + b/(2*s+3); pretty(F)

                                                b
                                        a/s + -------
                                              2 s + 3
pfd = cpf(q,F); pretty(pfd)

                                                6
                                        5/s - -------
                                              2 s + 3
f = ilaplace(pfd); pretty(f)

                               5 - 3 exp(- 3/2 t)
f = ilaplace(q); pretty(f) % check

                               5 - 3 exp(- 3/2 t)
%
echo off; diary off
```

Example F

Compute the inverse Laplace transform of
$Y(s) = \dfrac{7s^2+20s+53}{(s-1)(s^2+2s+5)}$

Solution

Same drill as before!

```
%
% NSS4-7.4/ Example F
%
syms a b c s
q = (7*s^2+20*s+53) / ((s-1)*(s^2+2*s+5)); pretty(q)

                                         2
                                      7 s  + 20 s + 53
                                   ----------------------
                                                2
                                   (s - 1) (s  + 2 s + 5)
F = a/(s-1) + (b*s+c)/(s^2+2*s+5); pretty(F)

                                      a        b s + c
                                    ----- + ------------
                                    s - 1     2
                                             s  + 2 s + 5
pfd = cpf(q,F); pretty(pfd)

                                     10        -3 s - 3
                                    ----- + ------------
                                    s - 1     2
                                             s  + 2 s + 5
%
%
```

```
f = ilaplace(pfd); pretty(f)

     10 exp(t) - 3 exp(-t) cos(2 t)
f = ilaplace(q); pretty(f) % check

     10 exp(t) - 3 exp(-t) cos(2 t)
%
echo off; diary off
```

Example G

Find the inverse Laplace transform of $Y(s) = \dfrac{s}{(s+2)^2(s^2+9)}$.

Solution

And this retires the side!

```
%
% NSS4-7.4/ Example G
%
syms a b c d s
q = s / ((s+2)^2*(s^2+9)); pretty(q)

                                       s
                               -----------------
                                      2   2
                               (s + 2)  (s  + 9)
F = a/(s+2) + b/(s+2)^2 + (c*s+d)/(s^2+9); pretty(F)

                           a        b       c s + d
                         ----- + -------- + -------
                         s + 2          2    2
                                 (s + 2)    s  + 9
pfd = cpf(q,F); pretty(pfd)

                                                         36
                                                - 5/169 s + ---
                    1             1                          169
           5/169 ----- - 2/13 -------- + -----------------
                 s + 2               2        2
                              (s + 2)        s  + 9
xpfd = expand(pfd); pretty(xpfd) % to aid in hand work

               1             1               s      36    1
      5/169 ----- - 2/13 -------- - 5/169 ------ + --- ------
            s + 2               2            2      169  2
                         (s + 2)            s  + 9       s  + 9
%
f = ilaplace(xpfd); pretty(f)

                                                    12
          (5/169 - 2/13 t) exp(-2 t) - 5/169 cos(3 t) + --- sin(3 t)
                                                    169
f = ilaplace(q); pretty(f) % check

                                                    12
          (5/169 - 2/13 t) exp(-2 t) - 5/169 cos(3 t) + --- sin(3 t)
                                                    169
%
echo off; diary off
```