

Fall 2003 Math 308/501–502

7 Laplace Transforms

7.5 Solving Initial Value Problems

Wed, 15/Oct ©2003, Art Belmonte

Summary

We now have all the pieces in place to solve initial value problems via Laplace transforms. There are four major steps involved (whether hand or machine techniques are used). If you are doing problems by hand, there may be several substeps involved in a given step. This will be illustrated in some of the hand examples.

Overview of the method

1. Take the Laplace transform of each side of the DE.
2. Substitute for the initial condition(s).
3. Solve for $Y(s)$, the Laplace transform of $y(t)$.
4. Take the inverse Laplace transform of $Y(s)$ to obtain $y(t)$.

Hand Examples

We'll start out with a very easy problem.

Example A1

Use Laplace transforms to solve the homogeneous IVP

$$y'' - 3y' - 4y = 0, \quad y(0) = 1, \quad y'(0) = -1$$

Solution

Here are the four steps mentioned in the overview of the method. Recall that $Y(s) = \mathcal{L}\{y(t)\}$.

1. Now $s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) - 4Y(s) = 0$.
2. Next, $s^2Y(s) - s + 1 - 3(sY(s) - 1) - 4Y(s) = 0$.
3. Thus $(s^2 - 3s - 4)Y(s) = s - 4$. Accordingly,
$$Y(s) = \frac{s - 4}{s^2 - 3s - 4} = \frac{s - 4}{(s + 1)(s - 4)} = \frac{1}{s + 1}$$
.
4. Therefore, $y(t) = e^{-t}$.

Example A2

For comparison, solve the same problem using Chapter 4 techniques. The amount of work involved is roughly the same in this instance.

Solution

- Now $r^2 - 3r - 4 = (r + 1)(r - 4) = 0$, whence $r = -1, 4$. Thus $y = c_1e^{-t} + c_2e^{4t}$ and $y' = -c_1e^{-t} + 4c_2e^{4t}$.
- The ICs give

$$1 = y(0) = c_1 + c_2 \quad \text{and} \quad -1 = y'(0) = -c_1 + 4c_2$$

Thus $0 = 5c_2$, whence $c_2 = 0$ and $c_1 = 1$.

- Therefore, $y = e^{-t}$.

Example B

Use Laplace transforms to solve the initial value problem

$$y' - 2y = e^{-t} \cos t, \quad y(0) = -2$$

Solution

Let's recount the steps in this more complicated example.

1. Take the Laplace transform of each side of the DE.

$$sY(s) - y(0) - 2Y(s) = \frac{(s + 1)}{(s + 1)^2 + 1^2}$$

2. Substitute for the initial condition(s).

$$sY(s) + 2 - 2Y(s) = \frac{s + 1}{s^2 + 2s + 2}$$

3. Solve for $Y(s)$, the Laplace transform of $y(t)$.

$$(s - 2)Y(s) = \frac{s + 1}{s^2 + 2s + 2} - 2$$
$$Y(s) = \frac{-2s^2 - 3s - 3}{(s - 2)(s^2 + 2s + 2)}$$

4. Take the inverse Laplace transform of $Y(s)$ to obtain $y(t)$.

- (a) And the sergeant gives the order that sends shivers down your spine: "*Fix bayonets!*" Yes, troops; it's time for some hand-to-hand combat. We must do a partial fraction decomposition of $Y(s)$.

- Now
$$\frac{-2s^2 - 3s - 3}{(s - 2)(s^2 + 2s + 2)} = \frac{a}{s - 2} + \frac{bs + c}{s^2 + 2s + 2}$$
.

- Thus
$$-2s^2 - 3s - 3 = a(s^2 + 2s + 2) + (bs + c)(s - 2)$$

or
$$-2s^2 - 3s - 3 = (a + b)s^2 + (2a - 2b + c)s + (2a - 2c)$$

- Equating coefficients leads to the linear system $a + b = -2$, $2a - 2b + c = -3$, $2a - 2c = -3$. We could solve this by hand by elimination and substitution. This, however, is not systematic. Instead, let's write it in matrix form.

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & -2 & 1 \\ 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ -3 \end{bmatrix} \text{ or } \mathbf{Ax} = \mathbf{r}$$

- We now have three ways we could proceed. All produce the same solution; namely, $a = -\frac{17}{10}$, $b = -\frac{3}{10}$, $c = -\frac{1}{5}$. (See the corresponding MATLAB example for all three ways!)
 - Row reduce the augmented matrix $\mathbf{M} = [\mathbf{A} \ \mathbf{b}]$ to row reduced echelon form.
 - Use `rref(M)` in MATLAB.
 - Do a linear solve via `A\b` in MATLAB.

(b) We now know that $Y(s) = \frac{-\frac{17}{10}}{s-2} + \frac{-\frac{3}{10}s - \frac{1}{5}}{(s^2 + 2s + 2)}$ or

$$Y(s) = -\frac{17}{10} \frac{1}{s-2} - \frac{3}{10} \frac{(s+1)}{(s+1)^2+1^2} + \frac{1}{10} \frac{1}{(s+1)^2+1^2}.$$

Taking the inverse Laplace transform gives

$$y(t) = -\frac{17}{10}e^{2t} - \frac{3}{10}e^{-t} \cos t + \frac{1}{10}e^{-t} \sin t$$

With the hand work that is actually involved in 4a-i, this has taken us a full column of work to solve this initial value problem. And they call this progress?! Clearly the toughest part was performing the partial fraction decomposition. Even taking the inverse Laplace transform wasn't that bad in comparison.

So instead of grinding through the partial fraction decomposition semiautomatically, we could use the `cpf` routine I wrote for you. All we need is to tell it the original rational function $Y(s)$ along the form of the decomposition $\frac{a}{s-2} + \frac{bs+c}{s^2+2s+2}$. This leads to the same solution for a, b, c . We'll try this together in MATLAB for practice.

Example C

Use Laplace transforms to solve the initial value problem

$$y'' - y' - 2y = t^2 e^{2t}, \quad y(0) = 0, \quad y'(0) = -1$$

Solution

- Take the Laplace transform of each side of the DE.

$$s^2 Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) - 2Y(s) = \frac{2}{(s-2)^3}$$

- Substitute for the initial condition(s).

$$s^2 Y(s) + 1 - sY(s) - 2Y(s) = \frac{2}{(s-2)^3}$$

- Solve for $Y(s)$, the Laplace transform of $y(t)$.

$$(s^2 - s - 2)Y(s) = \frac{2}{(s-2)^3} - 1$$

$$Y(s) = \frac{2 - (s-2)^3}{(s-2)^3(s^2 - s - 2)}$$

$$Y(s) = \frac{2 - (s-2)^3}{(s-2)^4(s+1)}$$

- Take the inverse Laplace transform of $Y(s)$ to obtain $y(t)$.

- This time we'll take the expressway and use `cpf`. The form of our partial fraction decomposition is

$$\frac{a}{s-2} + \frac{b}{(s-2)^2} + \frac{c}{(s-2)^3} + \frac{d}{(s-2)^4} + \frac{e}{s+1}$$

We obtain $[a, b, c, d, e] = \left[-\frac{29}{81}, \frac{2}{27}, -\frac{2}{9}, \frac{2}{3}, \frac{29}{81}\right]$. See the associated MATLAB example for details.

- Thus $Y(s) = -\frac{29}{81} \frac{1}{s-2} + \frac{2}{27} \frac{1}{(s-2)^2} - \frac{2}{9} \frac{1}{(s-2)^3} + \frac{2}{3} \frac{1}{(s-2)^4} + \frac{29}{81} \frac{1}{s+1}$. Taking the inverse Laplace transform yields

$$y(t) = \left(\frac{1}{9}t^3 - \frac{1}{9}t^2 + \frac{2}{27}t - \frac{29}{81}\right)e^{2t} + \frac{29}{81}e^{-t}$$

With the work involved in 4a to set up `cpf`, we're down to a half column of work to solve this initial value problem. This *is* progress. While the `cpf` routine certainly helps, we still needed to do some subsequent algebraic tweaking of the partial fractions to do the inverse Laplace transform table lookups.

MATLAB Examples

If we could just compute the inverse Laplace transform directly from the original form of $Y(s)$, we'd cut out a lot of work in Step 4. We can accomplish this by using `ilaplace`. We'll begin to bring more and more machine power to bear upon the tasks at hand. In our next example, we do everything by hand except the last step, in which we employ `ilaplace`. By now you should know the operations associated with the four steps.

Example D

Use Laplace transforms to solve the nonhomogeneous IVP

$$y'' + y = -2 \sin t, \quad y(0) = -1, \quad y'(0) = 1$$

Solution

- Now $s^2 Y(s) - sy(0) - y'(0) + Y(s) = -2 \frac{1}{s^2 + 1}$.
- Next, $s^2 Y(s) + s - 1 + Y(s) = \frac{-2}{s^2 + 1}$.
- Thus $(s^2 + 1)Y(s) = 1 - s - \frac{2}{s^2 + 1}$. Accordingly,

$$Y(s) = \frac{1-s}{s^2+1} - \frac{2}{(s^2+1)^2}$$
- Via **ilaplace**, $y(t) = (t - 1) \cos t$. Here is the snippet of code for just this step.

```
%
% NSS4-7.5/Example D
%
syms s
Y = (1-s) / (s^2+1) - 2 / (s^2+1)^2; pretty(Y);
          1 - s      2
          2      2      2
          s  + 1    (s  + 1)
y = ilaplace(Y); pretty(y)
          (-1 + t) cos(t)
%
echo off; diary off
```

Example C [revisited]

Here is the MATLAB work involved in “automatic” partial fraction decomposition via **cpf**. (The solution of the IVP is also verified via **ilaplace**.)

Solution

```
%
% NSS4-7.5/Example C
%
syms a b c d e s
Y = (2 - (s-2)^3) / ((s-2)^4 * (s+1)); pretty(Y)
          3
          2 - (s - 2)
          4
          (s - 2) (s + 1)
F = a/(s-2) + b/(s-2)^2 + c/(s-2)^3 + d/(s-2)^4 + e/(s+1);
pretty(F)
          a      b      c      d      e
          s - 2  2      3      4      s + 1
          (s - 2) (s - 2) (s - 2) (s - 2)
pdf = cpf(Y,F); pretty(pdf)
          29 1      1      1      1      29 1
          81 s - 2  + 2/27 (s - 2)  - 2/9 (s - 2)  + 2/3 (s - 2)  + 81 s + 1
%
%
%
f = ilaplace(pdf); pretty(f)
          / 29      2      3\      29
          | - -- + 2/27 t - 1/9 t + 1/9 t | exp(2 t) + -- exp(-t)
          \ 81      2      3 /      81
f = ilaplace(Y); pretty(f) % check
          / 29      2      3\      29
          | - -- + 2/27 t - 1/9 t + 1/9 t | exp(2 t) + -- exp(-t)
          \ 81      2      3 /      81
%
echo off; diary off
```

Example E

Use Laplace transforms to solve the nonhomogeneous IVP

$$y'' + 4y = e^{-2t} \cos t, \quad y(0) = 1, \quad y'(0) = 0$$

Solution

Finally, we bring full machine power to bear on *each* step of the IVP. The steps are the same ones we do by hand. This time, however, we use the following MATLAB commands to perform the computations.

- laplace**
- subs**
- solve**
- ilaplace**

Solution

Here is a diary file.

```
%
% NSS4-7.5/Example E
%
syms s t Ys
y = sym('y(t)');
de0 = diff(y,t,2) + 4*y - exp(-2*t)*cos(t);
pretty(de0)
          / 2      \
          |d      |
          |--- y(t) | + 4 y(t) - exp(-2 t) cos(t)
          | 2      |
          \dt     /
% 1
ltde = laplace(de0);
ltde = chiclet(ltde);
pretty(ltde)
          s + 2
          s (s Ys - y(0)) - Dy(0) + 4 Ys - -----
          2
          (s + 2) + 1
% 2
eq0 = subs(ltde, {'y(0)', 'Dy(0)'}, {1, 0});
pretty(eq0)
          s + 2
          s (s Ys - 1) + 4 Ys - -----
          2
          (s + 2) + 1
% 3
Ys = solve(eq0, Ys); pretty(Ys)
          3      2
          s + 4 s + 6 s + 2
          -----
          2      2
          (s + 4 s + 5) (s + 4)
% 4
y = ilaplace(Ys); pretty(y)
          7/65 exp(-2 t) cos(t) - 4/65 exp(-2 t) sin(t)
          + 58/65 cos(2 t) + 9/65 sin(2 t)
%
echo off; diary off
```

Example A1 [revisited]

Here is the MATLAB work involved in the “manual” partial fraction decomposition.

Solution

```

%
% T-263/8a
%
format rat
A = [1 1 0; 2 -2 1; 2 0 -2], r = [-2; -3; -3]
A =
    1     1     0
    2    -2     1
    2     0    -2
r =
   -2
   -3
   -3
M = [A r]
M =
    1     1     0    -2
    2    -2     1    -3
    2     0    -2    -3
%
% First way: Row operations, step-by-step.
%
M(2,:) = M(2,:) - M(2,1)*M(1,:)
M =
    1     1     0    -2
    0    -4     1     1
    2     0    -2    -3
M(3,:) = M(3,:) - M(3,1)*M(1,:)
M =
    1     1     0    -2
    0    -4     1     1
    0    -2    -2     1
%
M(2,:) = M(2,:) / M(2,2)
M =
    1     1     0    -2
    0     1    -1/4  -1/4
    0    -2     -2     1
M(1,:) = M(1,:) - M(1,2)*M(2,:)
M =
    1     0     1/4  -7/4
    0     1    -1/4  -1/4
    0    -2     -2     1
M(3,:) = M(3,:) - M(3,2)*M(2,:)
M =
    1     0     1/4  -7/4
    0     1    -1/4  -1/4
    0     0    -5/2   1/2
%
M(3,:) = M(3,:) / M(3,3)
M =
    1     0     1/4  -7/4
    0     1    -1/4  -1/4
    0     0     1    -1/5
M(1,:) = M(1,:) - M(1,3)*M(3,:)
M =
    1     0     0  -17/10
    0     1    -1/4  -1/4
    0     0     1    -1/5
M(2,:) = M(2,:) - M(2,3)*M(3,:)
M =
    1     0     0  -17/10
    0     1     0  -3/10
    0     0     1    -1/5
%
% Second way: Row operations via rref (all at once).
%
M = [A r]; MR = rref(M)
MR =
    1     0     0  -17/10
    0     1     0  -3/10
    0     0     1    -1/5

```

```

%
% Third way: Do a linear solve via A\b.
%
x = A\r
x =
   -17/10
   -3/10
   -1/5
%
echo off; diary off

```

Example F

Solve the third-order IVP

$$y''' + y'' + 3y' - 5y = 16e^{-t}, \quad y(0) = 0, \quad y'(0) = 2, \quad y''(0) = -4.$$

Solution

```

%
% NSS4-7.5/Example F
%
syms s t Ys
y = sym('y(t)');
de0 = diff(y,t,3) + diff(y,t,2) + 3*diff(y,t) ...
      - 5*y - 16*exp(-t);
pretty(de0)

```

$$\frac{d^3}{dt^3} y(t) + \frac{d^2}{dt^2} y(t) + 3 \frac{d}{dt} y(t) - 5 y(t) - 16 \exp(-t)$$

```

% 1
ltde = laplace(de0);
ltde = chiclet(ltde);
pretty(ltde)

```

$$s (s (s Ys - y(0)) - Dy(0)) - D2y(0) + s (s Ys - y(0)) - Dy(0) + 3 s Ys - 16 - 3 y(0) - 5 Ys - \frac{16}{s + 1}$$

```

% 2
eq0 = subs(ltde, {'y(0)', 'Dy(0)', 'D2y(0)'}, {0, 2, -4});
pretty(eq0)

```

$$s (s (s Ys - 2) + 2 + s Ys + 3 s Ys - 5 Ys - \frac{16}{s + 1})$$

```

% 3
Ys = solve(eq0, Ys); pretty(Ys)

```

$$\frac{2}{(s + 1) (s^2 + s + 3 s - 5)}$$

```

% 4
y = ilaplace(Ys); pretty(y)

```

$$-2 \exp(-t) + \exp(-t) \cos(2 t) + \exp(t)$$

```

%
echo off; diary off

```

Supplemental Examples

On Thu, 16/Oct/2003, I'll post two more examples regarding solving initial value problems with Laplace transforms. Look for these under Supplements on our class web page.