Spring 2006 Math 308-505
7 The Laplace Transform
7.8 Impulses and The Delta Function
Wed, 08/Mar ©2006, Art Belmonte

Summary

Construction of the delta function

Let $p \geq 0$. Recall from §7.6 the translated Heaviside function

$$u_p(t) = u(t-p) = \begin{cases} 0, & t < p; \\ 1 & t \geq p. \end{cases}$$

For $p \geq 0$, define $\delta_p^\epsilon(t) = \frac{1}{\epsilon}\left(u_p(t) - u_{p+\epsilon}(t)\right)$ or

$$\begin{cases} \frac{1}{\epsilon}, & \text{for } p \leq t < p+\epsilon; \\ 0 & \text{for } t < p \text{ or } t \geq p+\epsilon. \end{cases}$$

The **Dirac delta function** centered at p is defined as the limit

$$\delta_p(t) = \lim_{\epsilon \to 0} \delta_p^\epsilon(t).$$

We denote δ_0 by δ. NOTE THAT $\delta_p(t) = \delta(t-p)$. The delta "function" is an example of a *generalized function* or *distribution*. Colloquially speaking,

$$\delta(t) = \begin{cases} 0, & t \neq 0, \\ \infty, & t = 0. \end{cases}$$

Physically, it models a force that concentrates a great energy over a short duration, such as a hammer hitting a nail or a bat hitting a baseball. The delta function is known by its properties, which we now discuss.

Properties of the delta function

In the following p is a nonnegative constant, ϕ a function that is continuous near p, and f a piecewise continuous function. Moreover, we define $\delta * f = \lim_{\epsilon \to 0}\left(\delta_0^\epsilon * f\right)$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} \delta_p(t)\phi(t)\,dt = \int_{-\infty}^{\infty} \delta(t-p)\phi(t)\,dt &= \phi(p) \\ \mathcal{L}\left\{\delta_p(t)\right\} = \mathcal{L}\left\{\delta(t-p)\right\} &= e^{-ps} \\ \mathcal{L}\left\{\delta_0(t)\right\} = \mathcal{L}\left\{\delta(t)\right\} &= 1 \\ \frac{d}{dt}u(t-a) &= \delta(t-a) \\ f * \delta &= \delta * f = f \end{aligned}$$

(The first item is called the *sifting property*. The last item says that δ is the *identity* for the convolution product.)

Impulse response functions [revisited]

For constants a, b, c, the **(unit) impulse response function** is the solution $h(t)$ to the system represented by the symbolic initial value problem

$$ay'' + by' + cy = \delta, \quad y(0) = 0, \quad y'(0) = 0,$$

as our usual 4-step procedure shows.

1. $a(s^2Y(s) - sy(0) - y'(0)) + b(sY(s) - y(0)) + cY(s) = 1$
2. $(as^2 + bs + c)Y(s) = 1$
3. $Y(s) = 1/(as^2 + bs + c) = H(s)$, the transfer function;
4. $y(t) = h(t)$, the impulse response function from §7.7.

Hand Examples

412/4

Evaluate the integral $\int_{-\infty}^{\infty} e^{-2t}\delta(t+1)\,dt$.

Solution

The sifting property gives

$$\int_{-\infty}^{\infty} e^{-2t}\delta(t-(-1))\,dt = e^{-2(-1)} = e^2.$$

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Determine the Laplace transform of the generalized function $t^3\delta(t-3)$.

Solution

We have

$$\begin{aligned} \mathcal{L}\left\{t^3\delta(t-3)\right\} &= \int_0^{\infty} t^3\delta(t-3)e^{-st}\,dt \\ &= \int_{-\infty}^{\infty} t^3\delta(t-3)e^{-st}\,dt \\ &= 27e^{-3s} \end{aligned}$$

413/14

Solve the symbolic initial value problem

$$y'' + 2y' + 2y = \delta(t-\pi), \quad y(0) = 1, \quad y'(0) = 1,$$

then graph it for $0 \leq t \leq 6$.

Solution

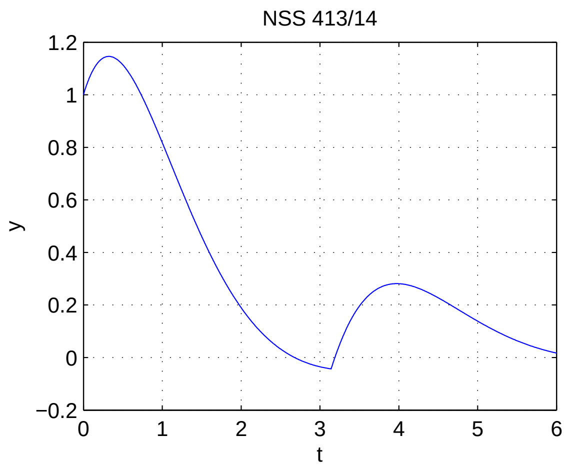
You know the drill.

1. $\left(s^2Y(s) - sy(0) - y'(0)\right) + 2(sY(s) - y(0)) + 2Y(s) = e^{-\pi s}$
2. $\left(s^2Y(s) - s - 1\right) + 2(sY(s) - 1) + 2Y(s) = e^{-\pi s}$
3. $\left(s^2 + 2s + 2\right) Y(s) = s + 3 + e^{-\pi s}$, whence

$$Y(s) = \frac{s + 3 + e^{-\pi s}}{(s+1)^2 + 1}$$

$$Y(s) = \frac{(s+1)}{(s+1)^2 + 1^2} + 2\frac{1}{(s+1)^2 + 1^2} + \frac{e^{-\pi s}}{(s+1)^2 + 1^2}$$

4. $y(t) = e^{-t}\cos t + 2e^{-t}\sin t + e^{-(t-\pi)}\sin(t-\pi)u(t-\pi)$
   or $y(t) = e^{-t}\cos t + 2e^{-t}\sin t - e^{-(t-\pi)}u(t-\pi)\sin t$



413/25

Find the impluse response function $y(t) = h(t)$ to the system

$$y'' + 4y' + 8y = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Solution

We have $h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{1/\left(s^2 + 4s + 8\right)\right\}$
$= \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{2}{(s+2)^2 + 2^2}\right\} = \frac{1}{2}e^{-2t}\sin 2t.$

411/Example 1

Solve and graph the solution of the IVP

$$y'' + 9y = 3\delta(t - \pi), \quad y(0) = 1, \quad y'(0) = 0.$$

Compare with graph of the solution of the IVP

$$y'' + 9y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution

While the theorem can't be applied to the first IVP (why?), our old drill still works!

1. $s^2Y(s) - sy(0) - y'(0) + 9Y(s) = 3e^{-\pi s}$
2. $s^2Y(s) - s + 9Y(s) = 3e^{-\pi s}$
3. $Y(s) = \dfrac{s + 3e^{-\pi s}}{s^2 + 9} = \dfrac{s}{s^2+9} + e^{-\pi s}\cdot\dfrac{3}{s^2+9}$
4. $f(t) = y(t) = \cos 3t + H(t-\pi)\sin(3(t-\pi))$

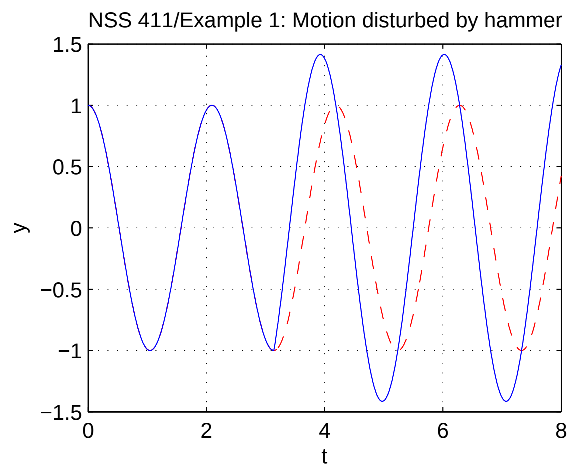
Reload and dispatch the second IVP.

1. $s^2Y(s) - sy(0) - y'(0) + 9Y(s) = 0$
2. $s^2Y(s) - s + 9Y(s) = 0$
3. $Y(s) = \dfrac{s}{s^2+9}$
4. $g(t) = y(t) = \cos 3t$

Mechanically, the second solution $g(t)$ represents simple harmonic motion, perhaps the displacement of a mass attached to an undamped spring. (Notice that in the DE $my'' + \mu y' + ky = 0$, we have $\mu = 0$.) This is the undisturbed system.

The first solution $f(t)$, on the other hand, represents what happens to the said mass-spring system when the mass is struck by a hammer that exerts an impulse on the mass at time $t = \pi$ seconds. This is the disturbed system.

The difference in the two systems is immediately discernible in this plot of the superimposed graphs!



MATLAB Examples

411/Example 1 [revisited]

Here is a MATLAB diary file showing the system solution and plot.

```
%
% NSS 411/Example 1
%
syms s t Ys
y = sym('y(t)')
y =
y(t)
de = diff(y,t,2) + 9*y - 3*dirac(t-pi);
pretty(de) % #0

                     / 2      \
                     |d       |
                     |--- y(t)| + 9 y(t) - 3 dirac(t - pi)
                     |  2     |
                     \dt      /
ltde = laplace(de); % #1
ltde = chiclet(ltde); pretty(ltde)

                 s (s Ys - y(0)) - Dy(0) + 9 Ys - 3 exp(-s pi)
eq = subs(ltde, {'y(0)' 'Dy(0)'}, {1 0}); % #2
pretty(eq)

                      s (s Ys - 1) + 9 Ys - 3 exp(-s pi)
Ys = solve(eq, Ys); pretty(Ys) % #3

                              s + 3 exp(-s pi)
                              ----------------
                                    2
                                   s  + 9
y = ilaplace(Ys); pretty(y) % #4

                     cos(3 t) - heaviside(t - pi) sin(3 t)
%
yh = dsolve('D2y + 9*y = 0', 'y(0)=1', 'Dy(0)=0', 't');
pretty(yh)

                                  cos(3 t)
t = linspace(0,8, 1000);
yh = eval(vectorize(yh));
plot(t,yh,'r--'); grid on; hold on
y = eval(vectorize(y));
plot(t,y)
xlabel('t'); ylabel('y')
title('NSS 411/Example 1: Motion disturbed by hammer')
%
echo off; diary off
```

413/14 [revisited]

Here is a MATLAB diary file showing the system solution and plot.

```
%
% NSS 413/14
%
syms s t Ys
y = sym('y(t)')
y =
y(t)
de = diff(y,t,2) + 2*diff(y,t) + 2*y - dirac(t-pi);
pretty(de) % #0

               / 2      \
               |d       |     /d      \
               |--- y(t)| + 2 |-- y(t)| + 2 y(t) - dirac(t - pi)
               |  2     |     \dt     /
               \dt      /
ltde = laplace(de); % #1
ltde = chiclet(ltde); pretty(ltde)

         s (s Ys - y(0)) - Dy(0) + 2 s Ys - 2 y(0) + 2 Ys - exp(-s pi)
eq = subs(ltde, {'y(0)' 'Dy(0)'}, {1 1}); % #2
pretty(eq)

                 s (s Ys - 1) - 3 + 2 s Ys + 2 Ys - exp(-s pi)
Ys = solve(eq, Ys); pretty(Ys) % #3

                             s + 3 + exp(-s pi)
                             ------------------
                                2
                               s  + 2 s + 2
y = ilaplace(Ys); pretty(y) % #4

   exp(-t) cos(t) + 2 exp(-t) sin(t) - heaviside(t - pi) exp(-t + pi) sin(t)
%
t = linspace(0,6,1000);
y = eval(vectorize(y));
plot(t,y); grid on
xlabel('t'); ylabel('y')
title('NSS 413/14')
%
echo off; diary off
```

412/4 [revisited]

We verify the integral we computed by hand.

```
%
% NSS 412/4
%
syms t
sift = int(exp(-2*t) * dirac(t+1), t, -inf, inf);
pretty(sift)

                                    exp(2)
%
echo off; diary off
```

413/10 [revisited]

We confirm our hand Laplace transform computation.

```
%
% NSS 413/10
%
syms t; syms s positive
sift = int(t^3 * dirac(t-3) * exp(-s*t), t, -inf, inf);
pretty(sift)

                                 27 exp(-3 s~)
syms s unreal
Lf = laplace(t^3 * dirac(t-3)); pretty(Lf)

                                 27 exp(-3 s)
%
echo off; diary off
```