Fall 2003 Math 308/501–502
9 Matrix Methods for Linear Systems
9.1 Introduction (to [Linear] Systems)
Wed, 05/Nov ©2003, Art Belmonte

Summary

General first-order systems

A general first-order system of n differential equations in the n unknown functions $x_1, x_2, \ldots, x_n$, has the **normal form**

$$\begin{aligned} x_1' &= g_1(t, x_1, x_2, \ldots, x_n) \\ x_2' &= g_2(t, x_1, x_2, \ldots, x_n) \\ &\cdots \\ x_n' &= g_n(t, x_1, x_2, \ldots, x_n). \end{aligned}$$

In vector form, $\mathbf{x}' = \mathbf{g}(t, \mathbf{x})$, where $\mathbf{x}' = \left[x_1'; x_2'; \ldots; x_n'\right]$ and $\mathbf{g}(t, \mathbf{x}) = \left[g_1(t, \mathbf{x}); g_2(t, \mathbf{x}); \ldots; g_n(t, \mathbf{x})\right]$ are $n \times 1$ *column* vectors. The independent variable is t. The dependent variables are $x_1, x_2, \ldots, x_n$. Or one could say the dependent *column* vector is $\mathbf{x} = [x_1; x_2; \ldots; x_n]$. In this general system, $\mathbf{g}$ may have components that contain *nonlinear* expressions involving the dependent variables $x_1, x_2, \ldots, x_n$.

Initial value problem

This consists of the vector differential equation $\mathbf{x}' = \mathbf{g}(t, \mathbf{x})$ together with the vector initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.

Converting higher-order equations to systems

Given the general nth order equation

$$y^{(n)} = g\left(t, y, y', y'', \cdots, y^{(n-1)}\right),$$

let $x_1 = y$, $x_2 = y'$, $x_3 = y''$,. . ., $x_n = y^{(n-1)}$. We thus have $x_k' = y^{(k)} = x_{k+1}$, $k = 1, 2, \ldots n-1$. Moreover,

$$x_n' = y^{(n)} = g\left(t, y, y', y'', \cdots, y^{(n-1)}\right) = g(t, x_1, x_2, \ldots, x_n).$$

Therefore, we have the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\cdots \\ x_{n-1}' &= x_n \\ x_n' &= g(t, x_1, x_2, \ldots, x_n). \end{aligned}$$

In other words, $\mathbf{x}' = \mathbf{g}(t, \mathbf{x})$, where $\mathbf{x}' = \left[x_1'; x_2'; \ldots; x_n'\right]$ and $\mathbf{g}(t, \mathbf{x}) = \left[x_2; x_3; \ldots; x_n; g(t, \mathbf{x})\right]$. The initial conditions $y^{(k)}(t_0) = y_k$, $k = 0, 1, \ldots, n-1$, become the vector initial condition $\mathbf{x}(t_0) = \mathbf{x}_0 = [y_0; y_1; \ldots; y_{n-1}]$.

Definition of a *linear* system; matrix notation

A **linear system** is one that may be written in the **normal form**

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t)$$

or more briefly, $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$. Here $\mathbf{x}$ is an $n \times 1$ *column* vector function, $\mathbf{A}$ is an $n \times n$ matrix function, and $\mathbf{f}$ is an $n \times 1$ *column* vector function (the **forcing term**). If $\mathbf{f} = \mathbf{0}$, the $n \times 1$ zero vector function, then the system is **homogeneous**; otherwise it is said to be **nonhomogeneous**.

The j-th row in this vector differential equation is

$$x_j' = f_j(t) + \sum_{k=1}^{n} a_{jk}(t) x_k(t).$$

The sum on the right-hand side consists of j-th element of the forcing term and the dot product of the j-th row of $\mathbf{A}$ with the column vector $\mathbf{x}$. (Can you say "matrix multiplication?" I knew you could. . .) Note that the x_k appear solely to the first power (hence the phrase "linear"). Moreover, while the a_{jk} and f_k functions may depend on the independent variable t (or be constants), they do *not* depend on the dependent variables (the x_k).

Hand Examples

Assume that the independent variable is t (time), unless stated otherwise.

Example A

Place the following system in the linear form $\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$ if possible, or explain why it is not possible.

$$\begin{aligned} \frac{1}{t} x_1' &= x_1 + (\cos t)\, x_2 \\ x_2' &= (\sin t)\, x_1 + 3x_2 \end{aligned}$$

Solution

Rewrite the system as

$$\begin{aligned} x_1' &= t x_1 + (t \cos t)\, x_2 \\ x_2' &= (\sin t)\, x_1 + 3x_2. \end{aligned}$$

Then we have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} t & t\cos t \\ \sin t & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{0}$, a linear homogeneous system.

Example B

Place the following system in the linear form $\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$ if possible, or explain why it is not possible.

$$\begin{aligned} x_1' &= -2x_1 + x_2^2 \\ x_2' &= 3x_1 - x_2 \end{aligned}$$

Solution

Due to the x_2^2 term, the system is *nonlinear*. Accordingly, it cannot be put into linear form!

Example C

Rewrite the following system using matrix notation.

$$\begin{aligned} x_1' &= -x_2 \\ x_2' &= x_1 \end{aligned}$$

Solution

We have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{0}$, a linear homogeneous system.

Example D

Rewrite this system using matrix notation.

$$\begin{aligned} x_1' &= -x_2 + \sin t \\ x_2' &= x_1 \end{aligned}$$

Solution

We have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \sin t \\ 0 \end{bmatrix},$$

or $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$, a linear nonhomogeneous system (since $\mathbf{f}(t) \neq \mathbf{0}$).

Example E

Convert the third-order differential equation and initial conditions

$$y''' + y'y'' = \sin \omega t, \quad y(0) = \alpha, \quad y'(0) = \beta, \quad y''(0) = \gamma$$

into a first-order system in vector notation.

Solution

Let $x_1 = y$, $x_2 = y'$, and $x_3 = y''$. Then

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = x_3 \\ x_3' &= y''' = \sin \omega t - y'y'' = \sin \omega t - x_2 x_3. \end{aligned}$$

In other words,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} x_2 \\ x_3 \\ \sin \omega t - x_2 x_3 \end{bmatrix}.$$

Therefore, $\mathbf{x}' = \mathbf{g}(t, \mathbf{x})$, where $\mathbf{x}' = \left[x_1'; x_2'; x_3'\right]$, $\mathbf{g}(t, \mathbf{x}) = [x_2; x_3; \sin \omega t - x_2 x_3]$, and $\mathbf{x}(0) = \mathbf{x}_0 = \left[\alpha; \beta; \gamma\right]$. This is a *nonlinear* system due to the $x_2 x_3$ term.

508/13

Express the following system of higher-order differential equations as a matrix system in normal form.

$$\begin{aligned} x'' - 3x' + t^2 y - (\cos t)x &= 0, \\ y''' + y'' - tx' + y' + e^t x &= 0. \end{aligned}$$

Solution

Let $u_1 = x$, $u_2 = x'$, $u_3 = y$, $u_4 = y'$, $u_5 = y''$. Then

$$\begin{aligned} u_1' &= x' = u_2, \\ u_2' &= x'' = 3x' - t^2 y + (\cos t)x = (\cos t)u_1 + 3u_2 - t^2 u_3, \\ u_3' &= y' = u_4, \\ u_4' &= y'' = u_5, \\ u_5' &= y''' = -e^t x + tx' - y' - y'' = -e^t u_1 + tu_2 - u_4 - u_5. \end{aligned}$$

In other words,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \cos t & 3 & -t^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -e^t & t & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}$$

i.e., $\mathbf{u}'(t) = \mathbf{A}(t)\mathbf{u}(t) + \mathbf{0}$ or $\mathbf{u}' = \mathbf{A}\mathbf{u} + \mathbf{0}$, a linear homogenous system with variable coefficients.