Fall 2003 Math 308/501–502
9 Matrix Methods for Linear Systems
9.2 Linear Algebraic Equations
Fri, 07/Nov ©2003, Art Belmonte

Summary

Linear system of *n* algebraic equations

This is a system in the n unknown functions $x_1, x_2, \ldots, x_n$, that has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + 2_{12}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n, \end{aligned}$$

where the a_{ij} and b_i are *constants*. In matrix-vector form, $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is an $n \times n$ constant matrix and $\mathbf{b}$ is $n \times 1$ *column* vector. The unknowns are $x_1, x_2, \ldots, x_n$. Or one could say the unknown *column* vector is $\mathbf{x} = [x_1; x_2; \ldots; x_n]$. Note that the system is *linear* since the unknowns only occur to the first power.

Gauss-Jordan elimination algorithm

This consists of systematically eliminating variables from one equation to the next, so as to be able to readily read off the solution. More precisely, it involves transforming the augmented matrix $\mathbf{M} = [\mathbf{A}, \mathbf{b}]$ to reduced row echelon form (RREF) via elementary row operations.

- Interchanging two rows of a matrix;
- Multiplying a row of a matrix by a nonzero scalar (a real [or complex] number);
- Adding a nonzero scalar multiple of one row to another row.

MATLAB Examples

You want hand examples? Take a linear algebra class. Here we employ MATLAB, which was literally invented to tackle these sorts of problems! (NOTE: **format rat** means rational format.)

512/2

Find all solutions to the system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 6 \\ 2x_1 + x_2 + x_3 &= 6 \\ x_1 + x_2 + 3x_3 &= 6. \end{aligned}$$

Solution

Rewrite the system in matrix-vector form as

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$$

or $\mathbf{Ax} = \mathbf{b}$. Form the augmented matrix $\mathbf{M} = [\mathbf{A}, \mathbf{b}]$

$$\begin{bmatrix} 1 & 2 & 2 & 6 \\ 2 & 1 & 1 & 6 \\ 1 & 1 & 3 & 6 \end{bmatrix},$$

then transform it to reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

For illustration, we'll do this three ways. That said, the "sure kill" is the second way: using the **rref** command in MATLAB.

1. The fastest way to solve a linear system with n equations in n unknowns—provided that it has a *unique* solution—is to left-divide the coefficient matrix $\mathbf{A}$ into the right-hand side vector $\mathbf{b}$; i.e., $\mathbf{x} = \mathbf{A} \backslash \mathbf{b}$.
2. A quick way that will *always* work is to use MATLAB's **rref** command. This is true in each of the three cases that may arise.
   (a) unique solution
   (b) infinitely many solutions
   (c) no solution
3. A slow way that mimicks hand solution is to use MATLAB to do the elementary row operations.

Here is a MATLAB diary file which shows these three techniques.

```
%
% NSS4-512/2
%
format rat
A = [1 2 2; 2 1 1; 1 1 3]
A =
      1           2           2
      2           1           1
      1           1           3
b = [6; 6; 6]
b =
      6
      6
      6
% Fastest way (if solution is unique)
x = A\b
x =
      2
      1
      1
%
%
```

```
% Quick way: sure kill; handles all 3 cases:
% 1. unique solution
% 2. infinitely many solutions
% 3. no solutions
M = [A b] % augmented matrix
M =
      1           2           2           6
      2           1           1           6
      1           1           3           6
MR = rref(M) % Reduced Row Echelon Form
MR =
      1           0           0           2
      0           1           0           1
      0           0           1           1
% Slow way (directly mimicking hand solution
% using row operations). This is rref by hand!
% Zero out first column below the diagonal.
M(2,:) = M(2,:) - 2*M(1,:); M
M =
      1           2           2           6
      0          -3          -3          -6
      1           1           3           6
M(3,:) = M(3,:) - M(1,:); M
M =
      1           2           2           6
      0          -3          -3          -6
      0          -1           1           0
% Zero out second column below the diagonal.
M(2,:) = M(2,:) / (-3); M
M =
      1           2           2           6
      0           1           1           2
      0          -1           1           0
M(1,:) = M(1,:) - 2*M(2,:); M
M =
      1           0           0           2
      0           1           1           2
      0          -1           1           0
M(3,:) = M(3,:) + M(2,:); M
M =
      1           0           0           2
      0           1           1           2
      0           0           2           2
% Zero out the third column below the diagonal.
M(3,:) = M(3,:) / 2; M
M =
      1           0           0           2
      0           1           1           2
      0           0           1           1
M(2,:) = M(2,:) - M(3,:); M
M =
      1           0           0           2
      0           1           0           1
      0           0           1           1
%
echo off; diary off
```

512/4

Find all solutions to the system

$$\begin{aligned} x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 1 \\ 2x_1 - x_2 + x_3 + 2x_4 &= 0 \\ 2x_1 - x_2 + x_3 + x_4 &= 0. \end{aligned}$$

Solution

It turns out that this system also has a unique solution. We illustrate using the first two (automatic) solution techniques.

```
%
% NSS4-512/4
%
format rat
A = [0 0 1 1; 1 1 1 1; 2 -1 1 2; 2 -1 1 1]
A =
      0           0           1           1
      1           1           1           1
      2          -1           1           2
      2          -1           1           1
b = [0; 1; 0; 0]
b =
      0
      1
      0
      0
% Fastest way (if solution is unique)
x = A\b
x =
     1/3
     2/3
      0
      0
% Quick way: sure kill; handles all 3 cases.
M = [A b] % augmented matrix
M =
      0           0           1           1           0
      1           1           1           1           1
      2          -1           1           2           0
      2          -1           1           1           0
MR = rref(M) % Reduced Row Echelon Form
MR =
      1           0           0           0          1/3
      0           1           0           0          2/3
      0           0           1           0           0
      0           0           0           1           0
%
echo off; diary off
```

512/6

Find all solutions to the system

$$\begin{aligned} -2x_1 + 2x_2 - x_3 &= 0 \\ x_1 - 3x_2 + x_3 &= 0 \\ 4x_1 - 4x_2 + 2x_3 &= 0. \end{aligned}$$

Solution

As we'll see, there are infinitely many solutions to this system. Notice that the fastest solution technique *fails* in this instance since there is *not* a unique solution. I include it here just so you know the sort of warning/error messages that occur.

```
%
% NSS4-512/6
% (NOTE: Once you have the rref of M, just
% use pencil and paper to finish off the problem!)
%
format rat
A = [-2 2 -1; 1 -3 1; 4 -4 2]
A =
     -2           2          -1
      1          -3           1
      4          -4           2
b = zeros(3,1)
b =
      0
      0
      0
```

```
% Fastest way (if solution is unique)
% BUT SOLUTION IS *NOT* UNIQUE IN THIS CASE!
x = A\b % "What we have here is a
Warning: Matrix is singular to working precision.
(Type "warning off MATLAB:singularMatrix" to suppress
this warning.)
> In /u/belmonte/2003c/308/T/c9/s2/p512x06/p512x06.m
at line 13
x =
     1/0
     1/0
     1/0
%         failure to communicate."
% Quick way: sure kill; handles all 3 cases.
M = [A b] % augmented matrix
M =
     -2            2           -1            0
      1           -3            1            0
      4           -4            2            0
MR = rref(M) % Reduced Row Echelon Form
MR =
      1            0          1/4            0
      0            1         -1/4            0
      0            0            0            0
% There are infinitely many solutions.
%
echo off; diary off
```

From the reduced row echelon form, we have $x_1 + \frac{1}{4}x_3 = 0$ and $x_2 - \frac{1}{4}x_3 = 0$. Letting $x_3 = s$, we have $x_1 = -\frac{1}{4}s$ and $x_2 = \frac{1}{4}s$. Equivalently, we could let $x_3 = 4s$. Then an FFF ("fraction-free formulation!") is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s \\ s \\ 4s \end{bmatrix}, \ s \in \mathbb{R} \text{ (or } \mathbb{C}\text{)}.$$

512/8

Find all solutions to the system

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -x_1 - x_2 + x_3 &= 0 \\ x_1 + x_2 - x_3 &= 0. \end{aligned}$$

Solution

Once again, there are infinitely many solutions to this system. This time we have a two-parameter set of solutions.

```
%
% NSS4-512/8
%
format rat
A = [1 1 -1; -1 -1 1; 1 1 -1]
A =
      1            1           -1
     -1           -1            1
      1            1           -1
b = zeros(3,1)
b =
      0
      0
      0
%
x = A\b
Warning: Matrix is singular to working precision.
(Type "warning off MATLAB:singularMatrix" to suppress
this warning.)
> In /u/belmonte/2003c/308/T/c9/s2/p512x08/p512x08.m
at line 10
x =
     1/0
     1/0
     1/0
%
M = [A b]
M =
      1            1           -1            0
     -1           -1            1            0
      1            1           -1            0
MR = rref(M)
MR =
      1            1           -1            0
      0            0            0            0
      0            0            0            0
```

Therefore $\quad x_1 = t - s, \quad x_2 = s, \quad x_3 = t, \quad s, t \in \mathbb{R}$ (or $\mathbb{C}$).

512/10

Find all solutions to the complex system

$$\begin{aligned} x_1 + x_2 + x_3 &= i \\ 2x_1 + 3x_2 - ix_3 &= 0 \\ x_1 + 2x_2 + x_3 &= i. \end{aligned}$$

Solution

Here is the unique complex solution dispatched via left division. Also note the default **format short** decimals.

```
%
% NSS4-512/10
%
A = [1 1 1; 2 3 -i; 1 2 1]
A =
   1.0000             1.0000             1.0000
   2.0000             3.0000                  0 - 1.0000i
   1.0000             2.0000             1.0000
b = [i; 0; i]
b =
        0 + 1.0000i
        0
        0 + 1.0000i
%
x = A\b
x =
  -0.4000 + 0.2000i
        0
   0.4000 + 0.8000i
%
echo off; diary off
```

512/12b

Find all solutions to the system

$$\begin{aligned} 2x_1 + x_3 &= -1 \\ -3x_1 + x_2 + 4x_3 &= 1 \\ -x_1 + x_2 + 5x_3 &= 1. \end{aligned}$$

Solution

The last row of the reduced row echelon form implies that $0 = 1$, a contradiction. Accordingly, there are *no* solutions to this linear system.

```
%
% NSS4-512/12b
%
format rat
A = [2 0 1; -3 1 4; -1 1 5]
A =
       2            0            1
      -3            1            4
      -1            1            5
b = [-1; 1; 1]
b =
      -1
       1
       1
%
M = [A b]
M =
       2            0            1           -1
      -3            1            4            1
      -1            1            5            1
MR = rref(M)
MR =
       1            0           1/2           0
       0            1          11/2           0
       0            0            0            1
%
echo off; diary off
```