# Fall 2003 Math 308/501–502
# 9 Matrix Methods for Linear Systems
# 9.4 Linear Systems in Normal Form
# Wed, 12/Nov ©2003, Art Belmonte

## *Summary*

### Definition of a linear system; matrix notation

A **linear system** is one that may be written in the **normal form**

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t)$$

or more briefly, $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$. Here $\mathbf{x}$ is an $n \times 1$ *column* vector function, $\mathbf{A}$ is an $n \times n$ matrix function, and $\mathbf{f}$ is an $n \times 1$ *column* vector function (the **forcing term**). If $\mathbf{f} = \mathbf{0}$, the $n \times 1$ zero vector function, then the system is **homogeneous**; otherwise it is said to be **nonhomogeneous**.

The $j$-th row in this vector differential equation is

$$x_j' = f_j(t) + \sum_{k=1}^{n} a_{jk}(t) x_k(t).$$

The sum on the right-hand side consists of $j$-th element of the forcing term and the dot product of the $j$-th row of $\mathbf{A}$ with the column vector $\mathbf{x}$. Note that the $x_k$ appear solely to the first power (hence the phrase “linear”). Moreover, while the $a_{jk}$ and $f_k$ functions may depend on the independent variable $t$ (or be constants), they do *not* depend on the dependent variables (the $x_k$).

### YALO (“Yet Another Linear Operator”)

Rewrite $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$ as $\mathbf{x}' - \mathbf{A}\mathbf{x} = \mathbf{f}$ and let $L[\mathbf{x}] = \mathbf{x}' - \mathbf{A}\mathbf{x}$. Then the system becomes $L[\mathbf{x}] = \mathbf{f}$. (NOTE: In this form, it's actually easier to see why the linear system is called homogeneous if $\mathbf{f} = \mathbf{0}$ and nonhomogeneous if $\mathbf{f} \neq \mathbf{0}$.) That $L$ is a *linear* operator follows immediately from the linearity of differentiation and matrix muliplication.

### Initial value problem for a linear system of ODEs

This consists of the differential equation $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$ together with the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.

### Existence and Uniqueness Theorem

With $\mathbf{A}(t)$ and $\mathbf{f}(t)$ defined as above and continuous on an interval $I$, let $t_0 \in I$ and $\mathbf{x}_0 \in \mathbb{R}^n$. Then the initial value problem

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

has a unique solution defined for *all* $t \in I$.

### Converting higher-order linear ODEs to systems

Given the $n$th order linear equation in standard form

$$y^{(n)}(t) + \sum_{j=0}^{n-1} p_j(t) y^{(j)}(t) = g(t),$$

let $x_1 = y$, $x_2 = y'$, $x_3 = y''$,. . ., $x_n = y^{(n-1)}$. We thus have $x_k' = y^{(k)} = x_{k+1}$, $k = 1, 2, \ldots n-1$. Moreover,

$$x_n' = y^{(n)} = g(t) - \sum_{j=0}^{n-1} p_j(t) y^{(j)}(t).$$

Therefore, we have the linear system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\cdots \\ x_{n-1}' &= x_n \\ x_n' &= g(t) - \sum_{j=1}^{n} p_j(t) y^{(j)}(t). \end{aligned}$$

In other words, $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$, where $\mathbf{x}' = \left[x_1'; x_2'; \ldots; x_n'\right]$, $\mathbf{f}(t) = [0; 0; \ldots; 0; g(t)]$, and $\mathbf{A}$ is the **companion matrix**

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -p_0(t) & -p_1(t) & -p_2(t) & \cdots & -p_{n-2}(t) & -p_{n-1}(t) \end{bmatrix}.$$

(All entries are zero except 1's along the superdiagonal and the $-p_j(t)$ along the bottom row.) The initial conditions become a vector initial condition; i.e. $y^{(k)}(t_0) = y_k$, $k = 0, 1, \ldots, n-1$, become $\mathbf{x}(t_0) = \mathbf{x}_0 = [y_0; y_1; \ldots; y_{n-1}]$.

### Linear (in)dependence and the Wronskian

The $m$ vector functions $\mathbf{x}_1(t), \ldots, \mathbf{x}_m(t)$ are **linearly dependent** on $I$ if there are constants $c_1, \ldots, c_m$, not all zero, such that $\sum_{j=1}^{m} c_j \mathbf{x}_j(t) = \mathbf{0}$, for all $t \in I$. They are **linearly independent** if $\sum_{j=1}^{m} c_j \mathbf{x}_j(t) = \mathbf{0}$, for all $t \in I$, implies the $c_j$'s are all zero.

Let $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ be $n \times 1$ column vector functions. We define the Wronskian (determinant) of this collection of $n$ vector functions to be the determinant of the square matrix whose columns are the $\mathbf{x}_k$.

If these $n$ vector functions $\mathbf{x}_k$ are *solutions* on an interval $I$ of the homogeneous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is an $n \times n$ matrix function that is continuous on $I$, then exactly one of following statements is true.

- The $\mathbf{x}_k$ are linearly dependent and their Wronskian is identically zero on $I$.
- The $\mathbf{x}_k$ are linearly independent and their Wronskian is never zero on $I$. In this case the $\mathbf{x}_k$ form a **fundamental set of solutions** of the homogeneous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Moreover, we call the matrix $\mathbf{X}$ whose columns are $\mathbf{x}_1, \ldots, \mathbf{x}_n$ a **fundamental matrix**.

### Representation of solutions (homogeneous case)

**Theorem** Let $\mathbf{X}$ be a fundamental matrix for the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is continuous on $I$. Then *every* solution of the system on $I$ may be represented as $\mathbf{x} = \mathbf{X}\mathbf{c} = \sum_{k=1}^{n} c_k \mathbf{x}_k$, where $\mathbf{X}$ is the $n \times n$ fundamental matrix (whose columns are the linearly independent solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$) and $\mathbf{c}$ is an $n \times 1$ *constant* column vector. We call $\mathbf{x} = \mathbf{X}\mathbf{c} = \sum_{k=1}^{n} c_k \mathbf{x}_k$ a **general solution** of the system.

### Representation of solutions (nonhomogeneous case)

**Theorem** With $\mathbf{A}$, $\mathbf{c}$, and $\mathbf{X}$ as in the preceding theorem, let $\mathbf{x}_p$ be a particular solution on the interval $I$ to the nonhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$. Then *every* solution of this system on $I$ may be represented as $\mathbf{x} = \mathbf{x}_p + \mathbf{X}\mathbf{c} = \mathbf{x}_p + \sum_{k=1}^{n} c_k \mathbf{x}_k$. We call this a **general solution** of the system.

## *Hand Examples*

Assume that the independent variable is $t$ (time), unless stated otherwise.

### 530/2

Write $\begin{cases} r'(t) = 2r(t) + \sin t \\ \theta'(t) = r(t) - \theta(t) + 1 \end{cases}$ in matrix form.

### Solution

We have $\begin{bmatrix} r \\ \theta \end{bmatrix}' = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} r \\ \theta \end{bmatrix} + \begin{bmatrix} \sin t \\ 1 \end{bmatrix}$, or $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$, a constant coefficient nonhomogeneous linear system.

### 530/3

Write $\begin{cases} dx/dt = t^2 x - y - z + t \\ dy/dt = e^t z + 5 \\ dz/dt = tx - y + 3z - e^t \end{cases}$ in matrix form.

### Solution

We have $\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} t^2 & -1 & -1 \\ 0 & 0 & e^t \\ t & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t \\ 5 \\ -e^t \end{bmatrix}$, or $\mathbf{u}' = \mathbf{A}\mathbf{u} + \mathbf{f}$, a variable coefficient nonhomogeneous linear system.

### 530/8

Rewrite the 3rd-order equation $y''' - y' + y = \cos t$ as a 1st-order system in matrix form.

### Solution

Let $x_1 = y$, $x_2 = y'$, $x_3 = y''$. Then

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = x_3 \\ x_3' &= y''' = -y + y' + \cos t = -x_1 + x_2 + \cos t \end{aligned}$$

or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cos t \end{bmatrix}$;i.e., $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$, a constant coefficient nonhomogeneous linear system. Note that $\mathbf{A}$ is a companion matrix.

### 531/12

Rewrite the system

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix} \mathbf{x} + t \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

as a set of scalar equations.

### Solution

We have $\begin{cases} x_1' = x_2 + t + 3 \\ x_2' = x_3 - t + 1 \\ x_3' = -x_1 + x_2 + 2x_3 + 2t. \end{cases}$

### 531/14

Determine whether $\mathbf{x}_1 = \begin{bmatrix} te^{-t} \\ e^{-t} \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$ are linearly dependent on $I = (-\infty, \infty)$.

### Solution

Assume that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly dependent. Then there exist constants $c_1$ and $c_2$, *not both zero*, such that $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{0}$, for all $t \in I$. In particular, this is true for $t = 0$, in which case $c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This implies that $c_2 = 0$ and $c_1 = -c_2 = 0$, a contradiction. Therefore, $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly *in*dependent.

### 531/20

Given that $\mathbf{x}_1 = e^{-t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{x}_2 = e^{4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are solutions to some linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, determine whether they form a fundamental solution set. If so, find a fundamental matrix for the system and give a general solution.

### Solution

By inspection, the matrix $\mathbf{A}$ is $2 \times 2$. Accordingly a fundamental solution set consists of two linearly independent solutions of the system. Let's compute the Wronskian determinant of $\mathbf{x}_1$ and $\mathbf{x}_2$.

$$\det \begin{bmatrix} 3e^{-t} & e^{4t} \\ 2e^{-t} & -e^{4t} \end{bmatrix} = -3e^{3t} - 2e^{3t} = -5e^{3t} \neq 0, \ \forall t \in \mathbb{R}.$$

Hence $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent. Thus a fundamental matrix is $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2] = \begin{bmatrix} 3e^{-t} & e^{4t} \\ 2e^{-t} & -e^{4t} \end{bmatrix}$ and a general solution is $\mathbf{x} = \mathbf{X}\mathbf{c}$, where $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is a constant column vector.

## *MATLAB Examples*

### 531/24

Verify that $\mathbf{x}_1 = e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = e^{-3t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ are solutions on $\mathbb{R}$ to the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, and that $\mathbf{x}_p = \begin{bmatrix} 5t+1 \\ 2t \\ 4t+2 \end{bmatrix}$ is a particular solution to the nonhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$, where $\mathbf{f} = t \begin{bmatrix} -9 \\ 0 \\ -18 \end{bmatrix}$. Finally, find a general solution of the nonhomogeneous system.

### Solution

MATLAB renders the needful, as the diary file below shows.

With $\mathbf{X} = [\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}]$ a fundamental matrix and $\mathbf{c} = [c_1; c_2; c_3]$ a constant vector, a general solution of the nonhomogeneous system is constructed as follows.

$$\mathbf{x} = \mathbf{x}_p + \mathbf{X}\mathbf{c} = \begin{bmatrix} 5t+1 \\ 2t \\ 4t+2 \end{bmatrix} + c_1 \begin{bmatrix} e^{3t} \\ 0 \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{3t} \\ e^{3t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -e^{-3t} \\ -e^{-3t} \\ e^{3t} \end{bmatrix}.$$

```
%
% NSS4-531/24
%
syms c1 c2 c3 t
A = [1 -2 2; -2 1 2; 2 2 1]
A =
     1    -2     2
    -2     1     2
     2     2     1
x1 = exp(3*t) * [1;0;1]

x1 =

[ exp(3*t)]
[        0]
[ exp(3*t)]

x2 = exp(3*t) * [-1;1;0]

x2 =

[ -exp(3*t)]
[  exp(3*t)]
[         0]

x3 = exp(-3*t) * [-1;-1;1]

x3 =

[ -exp(-3*t)]
[ -exp(-3*t)]
[  exp(-3*t)]

xp = [5*t+1; 2*t; 4*t+2]

xp =

[ 5*t+1]
[   2*t]
[ 4*t+2]

%
check1 = diff(x1,t) - A*x1

check1 =

[ 0]
[ 0]
[ 0]

check2 = diff(x2,t) - A*x2

check2 =

[ 0]
[ 0]
[ 0]

check3 = diff(x3,t) - A*x3

check3 =

[ 0]
[ 0]
[ 0]

X = [x1 x2 x3]

X =

[   exp(3*t),  -exp(3*t), -exp(-3*t)]
[          0,   exp(3*t), -exp(-3*t)]
[   exp(3*t),          0,  exp(-3*t)]

detX = det(X) % fundamental matrix
detX =
   3*exp(3*t)
%
c = [c1; c2; c3]

c =

[ c1]
[ c2]
[ c3]

x = xp + X*c; pretty(x) % general solution

  [5 t + 1 + exp(3 t) c1 - exp(3 t) c2 - exp(-3 t) c3]
  [                                                   ]
  [        2 t + exp(3 t) c2 - exp(-3 t) c3           ]
  [                                                   ]
  [      4 t + 2 + exp(3 t) c1 + exp(-3 t) c3         ]
%
echo off; diary off
```

### 532/28

In 532/26, a homework problem, you are asked to show that if $\mathbf{X}$ is a fundamental matrix for $\mathbf{x}' = \mathbf{A}\mathbf{x}$, then $\mathbf{x} = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0$ is the solution to the initial value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}, \ \mathbf{x}(t_0) = \mathbf{x}_0$.

Given this fact, first show that $\mathbf{X}(t) = \begin{bmatrix} e^{-t} & e^{5t} \\ -e^{-t} & e^{5t} \end{bmatrix}$ is a fundamental matrix for the system $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}\mathbf{x}$. Then find the solution of the system satisfying the initial condition $\mathbf{x}(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

## Solution

The unique solution is $\mathbf{x} = \begin{bmatrix} 2e^{-t} + e^{5t} \\ -2e^{-t} + e^{5t} \end{bmatrix}$. Here is a diary file.

```
%
% NSS4-532/28
%
format rat
syms t
A = [2 3; 3 2]
A =
      2           3
      3           2
X = [exp(-t) exp(5*t); -exp(-t) exp(5*t)]

X =

[  exp(-t), exp(5*t)]
[ -exp(-t), exp(5*t)]

x0 = [3;-1]
x0 =
      3
     -1
%
check1 = diff(X,t) - A*X % Columns are solutions.

check1 =

[ 0, 0]
[ 0, 0]

check2 = simple(det(X)) % They're linearly independent.

check2 =

2*exp(4*t)

%
Xinv = inv(X)

Xinv =

[  1/2/exp(-t), -1/2/exp(-t)]
[ 1/2/exp(5*t), 1/2/exp(5*t)]

Xinv0 = subs(Xinv, t, 0)
Xinv0 =
     1/2         -1/2
     1/2          1/2
x = X*Xinv0*x0; pretty(x) % unique solution

                            [2 exp(-t) + exp(5 t) ]
                            [                     ]
                            [-2 exp(-t) + exp(5 t)]
%
echo off; diary off
```