1. Determine whether \( S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 + x_2 = 0 \right\} \) is a subspace of \( \mathbb{R}^2 \). Prove your answer.

- \( S \) is nonempty since \( 0 \in S \).
- Let \( \alpha \in \mathbb{R} \) and \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S \). Then \( x_1 + x_2 = 0 \) and \( \alpha x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix} \) with \( \alpha x_1 + \alpha x_2 = \alpha (x_1 + x_2) = \alpha (0) = 0 \). So \( \alpha x \in S \). Therefore \( S \) is closed under scalar multiplication.
- Let \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) and \( y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \) be vectors in \( S \). Then \( x_1 + x_2 = 0 \) and \( y_1 + y_2 = 0 \). Therefore, \( x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} \) and \((x_1 + y_1) + (x_2 + y_2) = x_1 + x_2 + y_1 + y_2 = 0 + 0 = 0\).

Hence \( x + y \in S \). Thus \( S \) is closed under vector addition.
- Accordingly, \( S \) is a subspace of \( \mathbb{R}^2 \).

2. Let \( A = \begin{bmatrix} 1 & 3 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 5 & 5 & 5 \\ 0 & 0 & 4 & 4 & 4 \end{bmatrix} \).

(a) Find a basis for \( N(A) \), the null space of \( A \).

- The reduced row-echelon form of \( A \) is \( U = \begin{bmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \). Let \( x \in N(A) \).

Then \( x_1 + 3x_2 + 2x_4 + 3x_5 = 0 \) and \( x_3 + x_4 + x_5 = 0 \). So \( x \) has the form

\[
\begin{bmatrix} -3r - 2s - 3t \\ r \\ -s - t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \]

where the indicated \( v_1, v_2, v_3 \) form a basis for \( N(A) \), whose dimension is 3.

(b) Find a basis for the column space of \( A \). What is the rank of \( A \)?

- The first and third columns of \( U \) form a basis for the column space of \( U \). Since \( A \) is row equivalent to \( U \), the corresponding columns of \( A \) form a basis for the column space of \( A \); namely,

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}.

- The rank of \( A \) is the dimension of its row space. This is the number of nonzero rows in \( U \), the reduced row echelon form of \( A \), namely 2. (Recall from Section 3.6 that the row and column space of a matrix have the same dimension. Indeed, there are two vectors in the basis for the column space of \( A \) from the preceding item.)

3. Answer the following questions and, in each case, give geometric explanations for your answers.

(a) Is it possible to have a pair of one-dimensional subspaces \( U_1 \) and \( U_2 \) of \( \mathbb{R}^3 \) such that their intersection \( U_1 \cap U_2 \) is \( \{0\} \), the set consisting solely of the zero vector?

- Yes, it is possible. Let \( z \in U_1 \cap U_2 \) where

\[
U_1 = \left\{ v \in \mathbb{R}^3 : v = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix}, s \in \mathbb{R} \right\} \text{ and } U_2 = \left\{ w \in \mathbb{R}^3 : w = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}, t \in \mathbb{R} \right\}.
\]

Then \( \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} \). So \( s = t = 0 \) and thus \( z = 0 \). Therefore \( U_1 \cap U_2 = \{0\} \).

- Geometrically, \( U_1 \) is the \( x \)-axis and \( U_2 \) is the \( y \)-axis in 3-D space, where we identify position vectors with points. The origin is the intersection of these two straight lines.

(b) Is it possible to have a pair of two-dimensional subspaces \( V_1 \) and \( V_2 \) of \( \mathbb{R}^3 \) such that their intersection \( V_1 \cap V_2 \) is \( \{0\} \), the set consisting solely of the zero vector?

- No, it is not possible. Geometrically, a two-dimensional subspace of \( \mathbb{R}^3 \) is a plane that contains the origin. Two such planes intersect in a line through the origin, a one-dimensional subspace of \( \mathbb{R}^3 \), not \( \{0\} \), a zero-dimensional subspace.
4. Let \( \{u_1, u_2\} \) and \( \{v_1, v_2\} \) be ordered bases for \( \mathbb{R}^2 \), where
\[
\begin{align*}
  u_1 &= \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \\
  u_2 &= \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \\
  v_1 &= \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \\
  v_2 &= \begin{bmatrix} 4 \\ 9 \end{bmatrix}.
\end{align*}
\]
(a) Determine the transition matrix \( S \) from the standard basis \( \{e_1, e_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \) to the ordered basis \( \{u_1, u_2\} \). Use \( S \) to find the coordinates of \( x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) with respect to \( \{u_1, u_2\} \).

- In the nomenclature of Section 3.5 (q.v.), we have \( e = Uu \). So the transition matrix from \( \{u_1, u_2\} \) to \( \{e_1, e_2\} \) is
\[
U = [u_1, u_2] = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}.
\]
- Now \( Uu = e \) implies \( u = U^{-1}e \). So the transition matrix from \( \{e_1, e_2\} \) to \( \{u_1, u_2\} \) is \( S = U^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} \).
- The coordinates of \( x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) with respect to \( \{u_1, u_2\} \) are
\[
[x]_U = u = U^{-1}e = S[x]_E
= \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
= \begin{bmatrix} 5 \\ -2 \end{bmatrix}.
\]
- (You may optionally check this.)
\[
5u_1 - 2u_2 = 5 \begin{bmatrix} 1 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [x]_E
\]

(b) Determine the transition matrix \( T \) from the ordered basis \( \{v_1, v_2\} \) the ordered basis \( \{u_1, u_2\} \). Use \( T \) to find the coordinates of the vector \( z = 2v_1 + 3v_2 \) with respect to \( \{u_1, u_2\} \).

- In the nomenclature of Section 3.5 (q.v.), we have \( Uu = Vv \). Thus \( u = U^{-1}Vv \). So the transition matrix from \( \{v_1, v_2\} \) to \( \{u_1, u_2\} \) is \( T = U^{-1}V = SV \) or
\[
T = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 2 & 9 \end{bmatrix}
= \begin{bmatrix} 31 & 10 \\ -13 & -3 \end{bmatrix}.
\]
- The coordinates of \( z = 2v_1 + 3v_2 \) with respect to \( \{u_1, u_2\} \) are
\[
[z]_U = u = U^{-1}Vv = T[z]_V
= \begin{bmatrix} 31 & 10 \\ -13 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}
= \begin{bmatrix} 92 \\ -35 \end{bmatrix}.
\]
- (You may optionally check this.)
\[
2v_1 + 3v_2 = 2 \begin{bmatrix} 5 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 22 \\ 31 \end{bmatrix} = [z]_E
\]
\[
92u_1 - 35u_2 = 92 \begin{bmatrix} 1 \\ 3 \end{bmatrix} - 35 \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 22 \\ 31 \end{bmatrix} = [z]_E
\]

5. Let \( L \) be a linear operator on \( \mathbb{R}^2 \) and let \( v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), \( v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \), and \( v_3 = \begin{bmatrix} 1 \\ 7 \end{bmatrix} \). If \( L(v_1) = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \) and \( L(v_2) = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \), find the value of \( L(v_3) \).

- Since \( \mathbb{R}^2 \) has dimension 2 and \( v_1 \) and \( v_2 \) are linearly independent, \( \{v_1, v_2\} \) is an ordered basis for \( \mathbb{R}^2 \).
- Express \( v_3 \) as \( \alpha v_1 + \beta v_2 \) by solving a linear system. \[
\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 7 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 7 \end{bmatrix}
\]
Thus \( v_3 = 3v_1 + 2v_2 \), as you can verify.
- Since \( L \) is a linear operator, we have
\[
L(v_3) = L(3v_1 + 2v_2)
= 3L(v_1) + 2L(v_2)
= 3 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}
= \begin{bmatrix} 0 \\ 17 \end{bmatrix}.
\]

6. Let \( A \) and \( B \) be similar matrices. Show that \( \det(A) = \det(B) \).

- Since \( B \) is similar to \( A \), we have \( B = S^{-1}AS \) for some nonsingular matrix \( S \).
- Therefore,
\[
\det(B) = \det(S^{-1}AS)
= \det(S^{-1}) \det(A) \det(S)
= \det(S^{-1}) \det(S) \det(A)
= \det(S^{-1}S) \det(A)
= \det(I) \det(A)
= \det(A).
\]
- Hence \( \det(A) = \det(B) \).