1. Let \( x = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \) and \( y = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \).

(a) Find the vector projection \( p \) of \( x \) onto \( y \).

- The standard scalar (dot) product in \( \mathbb{R}^n \) is \( \langle u, v \rangle = u^T v \). The vector projection is

\[
 p = \frac{x - \langle x, y \rangle y}{\langle y, y \rangle} = \frac{\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \frac{3}{9} \begin{bmatrix} 5/3 \\ 2/3 \\ 4/3 \end{bmatrix}}{\frac{9}{3}} = \frac{\begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}}{2/3}. 
\]

(b) Verify that \( q = x - p \) is orthogonal to \( p \).

- Now \( q = \begin{bmatrix} 1/3 \\ 0 \\ 2 \end{bmatrix} \), whence \( \langle p, q \rangle = 0 \).

Thus \( p \) and \( q \) are orthogonal.

(c) Verify the Pythagorean Law holds for \( p, q, x \).

- The standard norm is \( \|v\| = \sqrt{\langle v, v \rangle} \). So

\[
\|p\|^2 + \|q\|^2 = 1^2 + 3^2 = (\sqrt{10})^2 = \|x\|^2.
\]

The Pythagorean Law is satisfied.

2. Let \( S \) be the 2-dimensional subspace of \( \mathbb{R}^3 \) spanned by \( x_1 \) and \( x_2 \), the columns of \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -2 \end{bmatrix} \).

(a) Find a basis for \( S^\perp \), the orthogonal complement of \( S \). Recall \( S^\perp = N(A^T) \).

- The reduced row echelon form of \( A^T \) is

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & -2
\end{bmatrix}.
\]

For \( v \in N(A^T) \) we have

\[
v_1 = -2v_3 \text{ and } v_2 = 2v_3.
\]

Accordingly, let

\[
v = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}.
\]

Then \( \{v\} \) is a basis for

\[
S^\perp = \text{Span}(v), \text{ a one-dimensional subspace of } \mathbb{R}^3.
\]

(b) Give a geometric description of \( S \) and \( S^\perp \).

- \( S \) is a plane through the origin in 3D space and \( S^\perp \) is a line through the origin that is perpendicular to this plane. [More precisely, the plane may be parametrically specified as \( tx_1 + ux_2, t, u \in \mathbb{R} \) or else

\[-2x + 2y + z = 0 \] using Cartesian coordinates since \( v \) is normal to the plane. The line is given by \( sv, s \in \mathbb{R}; \text{i.e., } x = -2s, y = 2s, z = s. \text{ See plot at end!} \]

3. Let \( A = \begin{bmatrix} 1 & -3 & -5 \\ 1 & 1 & -2 \\ 1 & -3 & 1 \\ 1 & 1 & 4 \end{bmatrix} \) and \( b = \begin{bmatrix} -6 \\ 1 \\ 1 \\ 6 \end{bmatrix} \).

(a) Use the QR command on your calculator to factor \( A = QR \) into the product of a matrix \( Q \) with orthonormal columns and an upper triangular matrix \( R \).

- We have \( Q = \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \) and

\[
R = \begin{bmatrix} 2 & -2 & -1 \\ 0 & 4 & 3 \\ 0 & 0 & 6 \end{bmatrix}.
\]

(b) Use the QR factorization to find the least squares solution of the system \( Ax = b \).

- The least squares solution is

\[
\hat{x} = R^{-1}Q^T b = \begin{bmatrix} 7/4 \\ 3/4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.75 \\ 0.75 \\ 1.00 \end{bmatrix}.
\]

4. Consider the vector space \( C[-1, 1] \) of continuous functions on \([-1, 1]\) with inner product defined by

\[
\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, dx.
\]

(a) Show that \( u_1(x) = \frac{1}{\sqrt{2}} \) and \( u_2(x) = \frac{\sqrt{6}}{2} x \) form an orthonormal set of vectors.

- Using the given inner product and its standard norm \( \|f\| = \sqrt{\langle f, f \rangle} \), we have

\[
\langle u_1, u_2 \rangle = 0, \|u_1\| = 1, \text{ and } \|u_2\| = 1.
\]

Hence \( u_1 \) and \( u_2 \) form an orthonormal set of vectors.

(b) Use part (a) to find the best least squares approximation to \( h(x) = x^3 - 3x + 2 \) by a linear function on \([-1, 1]\).

- We have \( c_1 = \langle h, u_1 \rangle = 2\sqrt{2} \) and \( c_2 = \langle h, u_2 \rangle = -\frac{\sqrt{6}}{2}. \)

The desired linear function is

\[
L(x) = c_1 u_1(x) + c_2 u_2(x) = 2 - \frac{\sqrt{6}}{2} x,
\]

a graph of which appears below along with that of \( h(x) \).
5. Let \( \mathbf{x} \) and \( \mathbf{y} \) be nonzero vectors in \( \mathbb{R}^n \) and \( \mathbf{Q} \) be an \( n \times n \) orthogonal matrix. If \( \mathbf{w} = \mathbf{Qx} \) and \( \mathbf{z} = \mathbf{Qy} \), show that the angle between \( \mathbf{w} \) and \( \mathbf{z} \) is equal to the angle between \( \mathbf{x} \) and \( \mathbf{y} \). Proceed as follows.

(a) Let \( \langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v} \) be the standard inner product on \( \mathbb{R}^n \). Show that \( \langle \mathbf{Qx}, \mathbf{Qy} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle \).

\[ \begin{align*}
\langle \mathbf{Qx}, \mathbf{Qy} \rangle &= (\mathbf{Qx})^T \mathbf{Qy} \\
&= \mathbf{x}^T \mathbf{Q}^T \mathbf{Qy} \\
&= \mathbf{x}^T \mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle
\end{align*} \]

because \( \mathbf{Q}^T \mathbf{Q} = \mathbf{I} \) since \( \mathbf{Q} \) is an orthogonal matrix.

(b) Let \( \| \mathbf{v} \| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \) be the standard norm on \( \mathbb{R}^n \). Note that \( \| \mathbf{Qx} \| = \| \mathbf{x} \| \) (and thus \( \| \mathbf{Qy} \| = \| \mathbf{y} \| \)).

From part (a), we have

\[ \| \mathbf{Qx} \| = \sqrt{\langle \mathbf{Qx}, \mathbf{Qx} \rangle} = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \| \mathbf{x} \|. \]

(c) Finally, show that angle between \( \mathbf{w} = \mathbf{Qx} \) and \( \mathbf{z} = \mathbf{Qy} \) is equal to the angle between \( \mathbf{x} \) and \( \mathbf{y} \).

- The angle between \( \mathbf{w} = \mathbf{Qx} \) and \( \mathbf{z} = \mathbf{Qy} \) is

\[ \theta = \cos^{-1} \left( \frac{\langle \mathbf{Qx}, \mathbf{Qy} \rangle}{\| \mathbf{Qx} \| \| \mathbf{Qy} \|} \right) = \cos^{-1} \left( \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\| \mathbf{x} \| \| \mathbf{y} \|} \right) \]

which is the angle between \( \mathbf{x} \) and \( \mathbf{y} \). Here we used the results from parts (a) and (b).

6. Let \( \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 2 & -2 \end{bmatrix} \).

(a) Find the eigenvalues of \( \mathbf{A} \) and corresponding eigenvectors.

- The characteristic polynomial of \( \mathbf{A} \) is

\[ p(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) = \lambda - \lambda^3 \]

via hand work or the \texttt{charPoly} command on your calculator.

- Eigenvalues of \( \mathbf{A} \) are the zeros of the characteristic polynomial: \( \lambda = -1, 0, 1 \). You may obtain these via factoring or by using the \texttt{cPolyRoots} command on your calculator.

- For \( \lambda = -1 \), the reduced row echelon form of \( \mathbf{A} - \lambda \mathbf{I} \) is \( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \). Thus

\[ u_1 = 0 \text{ and } u_2 = \frac{1}{2} u_3, \text{ whence } \mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \]

is an eigenvector associated with \( \lambda = -1 \).

- For \( \lambda = 0 \), the reduced row echelon form of \( \mathbf{A} - \lambda \mathbf{I} \) is \( \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \). Thus \( v_1 = 0 \) and \( v_2 = v_3 \), whence \( \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \) is an eigenvector associated with \( \lambda = 0 \).

- For \( \lambda = 1 \), the reduced row echelon form of \( \mathbf{A} - \lambda \mathbf{I} \) is \( \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \). Therefore,

\[ w_1 = w_2 = w_3, \text{ whence } \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]

is an eigenvector associated with \( \lambda = 1 \).

(b) Factor \( \mathbf{A} \) into a product \( \mathbf{XDX}^{-1} \) where \( \mathbf{X} \) is a nonsingular matrix and \( \mathbf{D} \) is a diagonal matrix.

- Using eigenvalues and eigenvectors from part (a), let \( \mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \) and

\[ \mathbf{X} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}. \]

- You may (optionally) verify on your calculator that the product \( \mathbf{XDX}^{-1} \),

\[ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ -1 & 2 & -1 \end{bmatrix}. \]

is equal to \( \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 2 & -2 \end{bmatrix} \).

#2: Plane with normal line at origin