1. Let \( \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) and \( \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \).

(a) Find the vector projection \( \mathbf{p} \) of \( \mathbf{x} \) onto \( \mathbf{y} \).

- The standard scalar (dot) product in \( \mathbb{R}^n \) is \( \langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v} \). The vector projection is

\[
\mathbf{p} = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle} \mathbf{y} = \frac{6}{18} \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 1/3 \\ 0 \end{bmatrix}.
\]

(b) Verify that \( \mathbf{q} = \mathbf{x} - \mathbf{p} \) is orthogonal to \( \mathbf{p} \).

- Now \( \mathbf{q} = \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \), whence \( \langle \mathbf{p}, \mathbf{q} \rangle = 0 \).

Thus \( \mathbf{p} \) and \( \mathbf{q} \) are orthogonal.

(c) Verify the Pythagorean Law holds for \( \mathbf{p}, \mathbf{q}, \mathbf{x} \).

- The standard norm is \( \| \mathbf{v} \| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \). So \( \| \mathbf{p} \|^2 + \| \mathbf{q} \|^2 = (\sqrt{2})^2 + (\sqrt{2})^2 = 2^2 = \| \mathbf{x} \|^2 \). The Pythagorean Law is satisfied.

2. Let \( S \) be the 2-dimensional subspace of \( \mathbb{R}^3 \) spanned by \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \), the columns of \( \mathbf{A} = \begin{bmatrix} 2 & 1 \\ 4 & 1 \\ 3 & 1 \end{bmatrix} \).

(a) Find a basis for \( S^\perp \), the orthogonal complement of \( S \). Recall \( S^\perp = N(\mathbf{A}^T) \).

- The reduced row echelon form of \( \mathbf{A}^T \) is

\[
\begin{bmatrix}
1 & 0 & 1/2 \\
0 & 1 & 1/2
\end{bmatrix}
\].

For \( \mathbf{v} \in N(\mathbf{A}^T) \) we have \( v_1 = -\frac{1}{2}v_3 \) and \( v_2 = -\frac{1}{2}v_3 \). Accordingly, let \( \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \). Then \( \{ \mathbf{v} \} \) is a basis for \( S^\perp = \text{Span} \{ \mathbf{v} \} \), a one-dimensional subspace of \( \mathbb{R}^3 \).

(b) Give a geometric description of \( S \) and \( S^\perp \).

- \( S \) is a plane through the origin in 3D space and \( S^\perp \) is a line through the origin that is perpendicular to this plane. [More precisely, the plane may be parametrically specified as \( t \mathbf{x}_1 + u \mathbf{x}_2, \ t, u \in \mathbb{R} \) or else \( x + y - 2z = 0 \) using Cartesian coordinates since \( \mathbf{v} \) is normal to the plane. The line is given by \( s \mathbf{v}, \ s \in \mathbb{R} \); i.e., \( x = s, \ y = s, \ z = -2s \). See plot at end!]

3. Let \( \mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \) and \( \mathbf{b} = \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix} \).

(a) Use the QR command on your calculator to factor \( \mathbf{A} = \mathbf{QR} \) into the product of a matrix \( \mathbf{Q} \) with orthonormal columns and an upper triangular matrix \( \mathbf{R} \).

- We have \( \mathbf{Q} = \begin{bmatrix} 2/3 & -\sqrt{2}/6 \\ 1/3 & \sqrt{2}/3 \\ 2/3 & -\sqrt{2}/6 \end{bmatrix} \) and

\[
\mathbf{R} = \begin{bmatrix} 3 & 5/3 \\ 0 & \sqrt{2}/3 \end{bmatrix}.
\]

(b) Use the QR factorization to find the least squares solution of the system \( \mathbf{A} \mathbf{x} = \mathbf{b} \).

- The least squares solution is \( \hat{\mathbf{x}} = \mathbf{R}^{-1} \mathbf{Q}^T \mathbf{b} = \begin{bmatrix} 9 \\ -3 \end{bmatrix} \).

4. Consider the vector space \( C[-1, 1] \) of continuous functions on \( [-1, 1] \) with inner product defined by \( \langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \, dx \).

(a) Show that \( u_1(x) = \frac{1}{\sqrt{2}} \) and \( u_2(x) = \frac{\sqrt{6}}{2} x \) form an orthonormal set of vectors.

- Using the given inner product and its standard norm \( \| f \| = \sqrt{\langle f, f \rangle} \), we have \( \langle u_1, u_2 \rangle = 0, \| u_1 \| = 1, \) and \( \| u_2 \| = 1 \). Hence \( u_1 \) and \( u_2 \) form an orthonormal set of vectors.

(b) Use part (a) to find the best least squares approximation to \( h(x) = x^3 - 2x \) by a linear function on \( [-1, 1] \).

- We have \( c_1 = \langle h, u_1 \rangle = 0 \) and \( c_2 = \langle h, u_2 \rangle = -\frac{7}{8} \sqrt{6} \). The desired linear function is

\[
L(x) = c_1 u_1(x) + c_2 u_2(x) = -\frac{7}{8} x,
\]

a graph of which appears below along with that of \( h(x) \).
5. Let \( \mathbf{x} \) and \( \mathbf{y} \) be nonzero vectors in \( \mathbb{R}^n \) and \( \mathbf{Q} \) be an \( n \times n \) orthogonal matrix. If \( \mathbf{w} = \mathbf{Qx} \) and \( \mathbf{z} = \mathbf{Qy} \), show that the angle between \( \mathbf{w} \) and \( \mathbf{z} \) is equal to the angle between \( \mathbf{x} \) and \( \mathbf{y} \). Proceed as follows.

(a) Let \((\mathbf{u}, \mathbf{v}) = \mathbf{u}^T \mathbf{v}\) be the standard inner product on \( \mathbb{R}^n \). Show that \( \langle \mathbf{Qx}, \mathbf{Qy} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle \).

- We have
  \[
  \langle \mathbf{Qx}, \mathbf{Qy} \rangle = (\mathbf{Qx})^T (\mathbf{Qy}) = \mathbf{x}^T \mathbf{Q}^T \mathbf{Qy} = \mathbf{x}^T \mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle
  \]
  because \( \mathbf{Q}^T \mathbf{Q} = \mathbf{I} \) since \( \mathbf{Q} \) is an orthogonal matrix.

(b) Let \( \| \mathbf{v} \| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \) be the standard norm on \( \mathbb{R}^n \). Note that \( \| \mathbf{v} \|^2 = \langle \mathbf{v}, \mathbf{v} \rangle \). Use (a) to show that \( \| \mathbf{Qx} \| = \| \mathbf{x} \| \) (and thus \( \| \mathbf{Qy} \| = \| \mathbf{y} \| \)).

- From part (a), we have
  \[
  \| \mathbf{Qx} \| = \sqrt{\langle \mathbf{Qx}, \mathbf{Qx} \rangle} = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \| \mathbf{x} \|.
  \]

(c) Finally, show that angle between \( \mathbf{w} \) and \( \mathbf{z} \) is equal to the angle between \( \mathbf{x} \) and \( \mathbf{y} \).

- The angle between \( \mathbf{w} = \mathbf{Qx} \) and \( \mathbf{z} = \mathbf{Qy} \) is
  \[
  \theta = \cos^{-1} \left( \frac{\langle \mathbf{Qx}, \mathbf{Qy} \rangle}{\| \mathbf{Qx} \| \| \mathbf{Qy} \|} \right)
  = \cos^{-1} \left( \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\| \mathbf{x} \| \| \mathbf{y} \|} \right),
  \]
  which is the angle between \( \mathbf{x} \) and \( \mathbf{y} \). Here we used the results from parts (a) and (b).

6. Let \( \mathbf{A} = \begin{bmatrix} 4 & -5 & 1 \\
1 & 0 & -1 \\
0 & 1 & -1 \end{bmatrix} \).

(a) Find the eigenvalues of \( \mathbf{A} \) and corresponding eigenvectors.

- The characteristic polynomial of \( \mathbf{A} \) is
  \[
  p(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) = -\lambda^3 + 3\lambda^2 - 2\lambda
  \]
  via hand work or the charPoly command on your calculator.

- Eigenvalues of \( \mathbf{A} \) are the zeros of the characteristic polynomial: \( \lambda = 0, 1, 2 \). You may obtain these via factoring or by using the cPolyRoots command on your calculator.

- For \( \lambda = 0 \), the reduced row echelon form of \( \mathbf{A} - \lambda \mathbf{I} \) is
  \[
  \begin{bmatrix} 1 & 0 & -1 \\
0 & 1 & -1 \\
0 & 0 & 0 \end{bmatrix}.
  \]
  Therefore,
  \[
  u_1 = u_2 = u_3, \text{ whence } \mathbf{u} = \begin{bmatrix} 1 \\
1 \\
1 \end{bmatrix}
  \]
  is an eigenvector associated with \( \lambda = 0 \).

- For \( \lambda = 1 \), the reduced row echelon form of \( \mathbf{A} - \lambda \mathbf{I} \) is
  \[
  \begin{bmatrix} 1 & 0 & -3 \\
0 & 1 & -2 \\
0 & 0 & 0 \end{bmatrix}.
  \]
  Thus
  \[
  v_1 = 3v_3 \text{ and } v_2 = 2v_3, \text{ whence } \mathbf{v} = \begin{bmatrix} 3 \\
2 \\
1 \end{bmatrix}
  \]
  is an eigenvector associated with \( \lambda = 1 \).

- For \( \lambda = 2 \), the reduced row echelon form of \( \mathbf{A} - \lambda \mathbf{I} \) is
  \[
  \begin{bmatrix} 1 & 0 & -7 \\
0 & 1 & -3 \\
0 & 0 & 0 \end{bmatrix}.
  \]
  Hence
  \[
  w_1 = 7w_3 \text{ and } w_2 = 3w_3, \text{ whence } \mathbf{w} = \begin{bmatrix} 7 \\
3 \\
1 \end{bmatrix}
  \]
  is an eigenvector associated with \( \lambda = 2 \).

(b) Factor \( \mathbf{A} \) into a product \( \mathbf{XDX}^{-1} \) where \( \mathbf{X} \) is a nonsingular matrix and \( \mathbf{D} \) is a diagonal matrix.

- Using eigenvalues and eigenvectors from part (a), let \( \mathbf{D} = \begin{bmatrix} 0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2 \end{bmatrix} \) and
  \[
  \mathbf{X} = \begin{bmatrix} 1 & 3 & 7 \\
1 & 2 & 3 \\
1 & 1 & 1 \end{bmatrix}.
  \]

- You may (optionally) verify on your calculator that the product \( \mathbf{XDX}^{-1} \),
  \[
  \begin{bmatrix} 1 & 3 & 7 \\
1 & 2 & 3 \\
1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & -2 & 5/2 \\
1/2 & -1 & 3 \\
1/2 & -1 & 1/2 \end{bmatrix},
  \]
  is equal to \( \mathbf{A} = \begin{bmatrix} 4 & -5 & 1 \\
1 & 0 & -1 \\
0 & 1 & -1 \end{bmatrix} \).