Summary

- The vectors \( v_1, \ldots, v_n \) form a basis for a vector space \( V \) if and only if \( v_1, \ldots, v_n \) are both linearly independent and a spanning set for \( V \).

- If \( \{v_1, \ldots, v_n\} \) is a spanning set for a vector space \( V \), then any collection of \( m \) vectors in \( V \) with \( m > n \) is linearly dependent.

- If a vector space \( V \) has a basis that consists of \( n \) vectors, it is said to be a finite dimensional vector space of dimension \( n \); otherwise it is infinite dimensional. See pages 146–147 for geometrical interpretations. [The zero subspace \( \{0\} \) is said to have dimension 0 (zero).]

- If \( \{v_1, \ldots, v_n\} \) and \( \{u_1, \ldots, u_m\} \) are bases for a vector space, then \( m = n \). That is, each basis of a finite dimensional vector space has the same number of elements.

- If a vector space \( V \) has dimension \( n > 0 \), then
  - 1. any set of \( n \) linearly independent vectors spans \( V \);
  - 2. any \( n \) vectors which span \( V \) are linearly independent.

- If a vector space \( V \) has dimension \( n > 0 \), then
  - 1. no set of fewer than \( n \) vectors spans \( V \);
  - 2. any set of fewer than \( n \) linearly independent vectors can be extended to form a basis for \( V \);
  - 3. any spanning set of \( V \) which contains more than \( n \) vectors can be pared down to form a basis for \( V \).

Examples

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Consider the vectors

\[
x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}.
\]

(a) Show that \( x_1 \) and \( x_2 \) form a basis for \( \mathbb{R}^2 \).

(b) Why must \( x_1, x_2, x_3 \) be linearly dependent?

(c) What is the dimension of \( \text{Span}(x_1, x_2, x_3) \)?

Solution

(a) Let \( A = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \) be the matrix whose columns are \( x_1 \) and \( x_2 \).

- Since \( \det(A) = 6 - 4 = 2 \neq 0 \), the columns of \( A \), \( x_1 \) and \( x_2 \), are linearly independent.

- Let \( b \in \mathbb{R}^2 \). Since \( \det(A) \neq 0 \), the matrix \( A \) is nonsingular. Hence the linear system \( Ax = b \) has the unique solution \( x = A^{-1}b = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \). Thus \( x \) is a linear combination \( w_1 x_1 + w_2 x_2 \) of the columns of \( A \). Since \( b \) was arbitrary, we see \( x_1 \) and \( x_2 \) span \( \mathbb{R}^2 \).

- Inasmuch as \( x_1 \) and \( x_2 \), are linearly independent and span \( \mathbb{R}^2 \), they form a basis for \( \mathbb{R}^2 \).

(b) Since \( x_1 \) and \( x_2 \) span \( \mathbb{R}^2 \), \( x_3 \) is a linear combination of \( x_1 \) and \( x_2 \). So vectors \( x_1, x_2, x_3 \) are linearly dependent.

(c) In light of part (b), the dimension of \( \text{Span}(x_1, x_2, x_3) \) is the same as \( \text{Span}(x_1, x_2) = \mathbb{R}^2 \), which is 2.

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Find a basis for the subspace \( S \) of \( \mathbb{R}^4 \) consisting of all vectors of the form \( \begin{bmatrix} a+b & a-b+2c & b & c \end{bmatrix}^T \), where \( a, b, c \in \mathbb{R} \). What is the dimension of \( S \)?

Solution

(a) Let \( M = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \) be an augmented homogeneous system matrix whose columns are \( v_1, v_2, v_3 \), along with the \( 4 \times 1 \) zero column vector. The reduced row echelon form of \( M \) is

\[
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

There are no free variables. The only solution is \( k = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \), signifying that \( v_1, v_2, v_3 \) are linearly independent.

- Let \( v \in S \). Since \( v = av_1 + bv_2 + cv_3 \) is equal to \( \begin{bmatrix} a+b & a-b+2c & b & c \end{bmatrix}^T \), we see that \( v_1, v_2, v_3 \) span \( S \).

- Therefore \( v_1, v_2, v_3 \) form a basis for \( S \).
Let $S$ be the subspace of $P_3$ consisting of all polynomials of degree $< 3$ of the form $ax^2 + bx + 2a + 3b$. Determine a basis for $S$.

**Solution**

- Let $v_1 = x^2 + 2$ and $v_2 = x + 3$, obtained by partially differentiating the stated form by $a, b$, respectively.

- If $c_1v_1 + c_2v_2 = z(x)$, the zero polynomial, then $c_1x^2 + c_2x + (2c_1 + 3c_2) = 0x^2 + 0x + 0$ for all $x$. Equating like coefficients, we have $c_1 = 0$ and $c_2 = 0$. Hence $v_1$ and $v_2$ are *linearly independent*.

- Let $v \in S$. Since $v = av_1 + bv_2 = ax^2 + bx + 2a + 3b$, we see that $v_1$ and $v_2$ span $S$.

- Therefore $v_1$ and $v_2$ form a *basis* for $S$.

In $C [-\pi, \pi]$, find the dimension of the subspace spanned by $1, \cos 2x, \cos^2 x$.

**Solution**

Recall from trigonometry that $\cos 2x = 2\cos^2 x - 1$. Hence the span of the three expressions is the same as that spanned by the pair $1$ and $\cos^2 x$. The pair is linearly independent since neither function is a constant multiple of the other. Thus the dimension of this pair and hence that of $1, \cos 2x, \cos^2 x$ is 2.