Math 311: Topics in Applied Math 1
3: Vector Spaces
3.5: Change of Basis

Summary
Pay special attention to the typography!

The Framework

- Let $V$ be a vector space of dimension $n$ over a field $F$ of scalars and let $v \in V$ be any vector in $V$.

- If $U = \{u_1, \ldots, u_n\}$ is an ordered basis for $V$, then $v$ can be written uniquely as a linear combination $v = \sum_{k=1}^{n} u_k v_k$ of the basis vectors, where $u_k \in F$.

- Column vector $u = [u_1, \ldots, u_n]^T$ is the coordinate vector of $v$ with respect to the ordered basis $U$, denoted by $u = [v]_U$. The scalars $u_k$ are called the coordinates of $v$ relative to $U$.

- If $W = \{w_1, \ldots, w_n\}$ is another basis for $V$, then $v = \sum_{k=1}^{n} w_k v_k$ with $w_k \in F$. So $w = [w_1, \ldots, w_n]^T$ is the coordinate vector $w = [v]_W$ of $v$ with respect to $W$, with $w_k$ the coordinates of $v$ relative to $W$.

- Let $U = [u_1, \ldots, u_n]$ and $W = [w_1, \ldots, w_n]$ be the $n \times n$ matrices whose columns consist of the elements of the sets $U$ and $W$, respectively. They are nonsingular since $U$ and $W$ are bases for $V$. Moreover, $Ww = Uu$, since each side represents $v$ as a linear combination of basis vectors.

The General Case

1. Now $Ww = Uu$ implies $w = (W^{-1}U)u = Tu$, where $T = W^{-1}U$ is the transition matrix from the basis $U$ to the basis $W$. It converts the $u = [v]_U$ coordinate representation of $v$ to the $w = [v]_W$ one.

2. Similarly, $Uu = Ww$ gives $u = (U^{-1}W)w = T^{-1}w$, converting the $w = [v]_W$ coordinate representation of $v$ to the $u = [v]_U$ one via the transition matrix $T^{-1} = U^{-1}W$.

A Special Case (deduced from General Case)

- Recall $E = \{e_1, \ldots, e_n\}$, the standard basis for $V = \mathbb{R}^n$, where $e_k$ is an $n$-element column vector with a 1 in the $k$th row and zeros elsewhere. For $v \in V$ we have $v = \sum_{k=1}^{n} e_k v_k$, with $e_k \in F = \mathbb{R}$. So $e = [e_1, \ldots, e_n]^T$ is the coordinate vector $e = [v]_E$ of $v$ with respect to the standard basis $E$, with $e_k$ the coordinates of $v$ relative to $E$.

- Let $E = [e_1, \ldots, e_n]$ be the $n \times n$ matrix whose columns consist of the elements of the set $E$. Note that $E = I$, the $n \times n$ identity matrix. This is what makes it a special case!

- Finally, let $v \in V$, $w = [v]_W$, and $W = [w_1, \ldots, w_n]$ be as specified in the Framework above. Therefore, $Ww = Ie$, since each side represents $v$ as a linear combination of basis vectors.

1. Now $Ww = Ie$ implies $w = W^{-1}Ie = W^{-1}e$, so $w = W^{-1}e = Se$, where $S = W^{-1}$ is the (special) transition matrix from basis $E$ to basis $W$. It converts the $e = [v]_E$ coordinate representation of $v$ to the $w = [v]_W$ one.

2. Also, $Ie = Ww$ gives $e = I^{-1}Ww = IWw = Ww$, so $e = Ww = S^{-1}w$, converts the $w = [v]_W$ coordinate representation of $v$ to the $e = [v]_E$ one via the (special) transition matrix $S^{-1} = W$.

Advisory
Study the examples below as well as those given in the calculator video for this section. Executive Summaries show how things can be done quickly IF:

1. You understand precisely what you’re doing.
2. You use massive firepower (your calculator).

Afterward, we’ll redo the examples with checks on our work so that we gain confidence in what we’re doing.

Executive Summaries

159/1b
Find the transition matrix corresponding to the change of basis from $U = \{u_1, u_2\}$ to $E = \{e_1, e_2\}$ for $\mathbb{R}^2$. Here

$u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. 

Solution

In the context of the Summary, we have \( \mathbf{Ie} = \mathbf{Uu} \), whence \( \mathbf{e} = \mathbf{Uu} \). So the desired transition matrix is \( \mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2] = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \).

159/2b

Find the transition matrix corresponding to the change of basis from \( E = \{\mathbf{e}_1, \mathbf{e}_2\} \) to \( U = \{\mathbf{u}_1, \mathbf{u}_2\} \) for \( \mathbb{R}^2 \). Here

\[
\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix},
\]

as in 159/1b.

Solution

In the manner of the Summary, we have \( \mathbf{Uu} = \mathbf{Ie} \), whence \( \mathbf{u} = \mathbf{U}^{-1}\mathbf{e} \). So the desired transition matrix is

\[
\mathbf{U}^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix},
\]

computed via calculator.

159/3b

Find the transition matrix corresponding to the change of basis from \( W = \{\mathbf{w}_1, \mathbf{w}_2\} \) to \( U = \{\mathbf{u}_1, \mathbf{u}_2\} \) for \( \mathbb{R}^2 \). Here

\[
\mathbf{w}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix},
\]

(\( U \) as in 159/2b). Note standard basis representations!

Solution

We have \( \mathbf{Uu} = \mathbf{Ww} \), whence \( \mathbf{u} = (\mathbf{U}^{-1}\mathbf{W}) \mathbf{w} \). So the desired transition matrix is

\[
\mathbf{U}^{-1}\mathbf{W} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 14 \\ -4 & -5 \end{bmatrix}.
\]

Examples with Checks

159/1b

Find the transition matrix corresponding to the change of basis from \( U = \{\mathbf{u}_1, \mathbf{u}_2\} \) to \( E = \{\mathbf{e}_1, \mathbf{e}_2\} \) for \( \mathbb{R}^2 \). Here

\[
\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

Solution

- In the context of the Summary, we have \( \mathbf{Ie} = \mathbf{Uu} \), whence \( \mathbf{e} = \mathbf{Uu} \). So the desired transition matrix is \( \mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2] = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \).

- Don’t take my word for it. Let’s check! First map the basis vectors \( U \), then an arbitrary vector in \( \mathbb{R}^2 \).

  - Let \( \mathbf{v} = \mathbf{u}_1 = 1\mathbf{u}_1 + 0\mathbf{u}_2 \). Hence \( \mathbf{e} = [\mathbf{v}]_U = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \).

  - Thus \( [\mathbf{v}]_E = \mathbf{Uu} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \), the coordinates of \( \mathbf{v} = \mathbf{u}_1 \) with respect to basis \( E \).

  - Let \( \mathbf{v} = \mathbf{u}_2 = 0\mathbf{u}_1 + 1\mathbf{u}_2 \). Hence \( \mathbf{e} = [\mathbf{v}]_U = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \).

  - Thus \( [\mathbf{v}]_E = \mathbf{Uu} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \), the coordinates of \( \mathbf{v} = \mathbf{u}_2 \) with respect to basis \( E \).

  - Finally, let \( \mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 \) be an arbitrary vector in \( \mathbb{R}^2 \). Then \( \mathbf{e} = [\mathbf{v}]_U = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \). Accordingly,

\[
[\mathbf{v}]_E = \mathbf{Uu} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1c_1 + 2c_2 \\ 2c_1 + 5c_2 \end{bmatrix} \mathbf{e} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = [\mathbf{v}]_U,
\]

whence \( \mathbf{e} = [\mathbf{v}]_E \) really does represent \( \mathbf{v} \).

- For future reference, it suffices (if desired) to just check that the basis vectors are mapped correctly.

159/2b

Find the transition matrix corresponding to the change of basis from \( E = \{\mathbf{e}_1, \mathbf{e}_2\} \) to \( U = \{\mathbf{u}_1, \mathbf{u}_2\} \) for \( \mathbb{R}^2 \). Here

\[
\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix},
\]

as in 159/1b.

Solution

- In the manner of the Summary, we have \( \mathbf{Uu} = \mathbf{Ie} \), whence \( \mathbf{u} = \mathbf{U}^{-1}\mathbf{e} \). So the desired transition matrix is

\[
\mathbf{U}^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix},
\]

computed via calculator.

- Once more for old time’s sake, let’s (just) check that the basis vectors \( E \) are mapped correctly.
Let $v = e_1 = 1e_1 + 0e_2$. So $e = [v]_E = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]$ and $[v]_U = u = U^{-1}e = \left[ \begin{array}{cc} 5 & -2 \\ -2 & 1 \end{array} \right] \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] = \left[ \begin{array}{c} 5 \\ -2 \end{array} \right]$. So $5u_1 - 2u_2 = 5 \left[ \begin{array}{c} 1 \\ 2 \end{array} \right] - 2 \left[ \begin{array}{c} 2 \\ 5 \end{array} \right] = \left[ \begin{array}{c} 5 - 4 \\ 10 - 10 \end{array} \right]$, or $\left[ \begin{array}{c} 1 \\ 0 \end{array} \right] = [v]_E$ as required.

Let $v = e_2 = 0e_1 + 1e_2$. So $e = [v]_E = \left[ \begin{array}{c} 0 \\ 1 \end{array} \right]$ and $[v]_U = u = U^{-1}e = \left[ \begin{array}{cc} 5 & -2 \\ -2 & 1 \end{array} \right] \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] = \left[ \begin{array}{c} -2 \\ 1 \end{array} \right]$. So $-2u_1 + 1u_2 = -2 \left[ \begin{array}{c} 1 \\ 2 \end{array} \right] + 1 \left[ \begin{array}{c} 2 \\ 5 \end{array} \right] = \left[ \begin{array}{c} -2 + 2 \\ -4 + 5 \end{array} \right]$, or $\left[ \begin{array}{c} 0 \\ 1 \end{array} \right] = [v]_E$ as required.

The calculator made things easy in the first item of the solution since it can compute an inverse matrix automatically. You simply type $U^{-1}$. Here's how to do it by hand.

From the Section 1.5 Summary, to compute $U^{-1}$, we transform the augmented matrix $[U | I]$ into its reduced row echelon form, $[I | U^{-1}]$, the right partition of which is $U^{-1}$. You may do this via elementary row operations by hand (or better yet with the calculator). Alternatively, employ the rref command for a fully automatic experience.

\[
\left[ \begin{array}{ccc} 1 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{array} \right]
\]

159/3b

Find the transition matrix corresponding to the change of basis from $W = \{w_1, w_2\}$ to $U = \{u_1, u_2\}$ for $\mathbb{R}^2$. Here $w_1 = \left[ \begin{array}{c} 3 \\ 2 \end{array} \right], \quad w_2 = \left[ \begin{array}{c} 4 \\ 3 \end{array} \right], \quad u_1 = \left[ \begin{array}{c} 1 \\ 2 \end{array} \right], \quad u_2 = \left[ \begin{array}{c} 2 \\ 5 \end{array} \right], \quad (U \text{ as in 159/2b}).$ Note standard basis representations!

Solution

We have $Uu = Ww$, whence $u = (U^{-1}W) w$. So the desired transition matrix is

\[
U^{-1}W = \left[ \begin{array}{cc} 1 & 2 \\ 2 & 5 \end{array} \right]^{-1} \left[ \begin{array}{ccc} 3 & 4 \\ 2 & 3 \end{array} \right] = \left[ \begin{array}{cc} 5 & -2 \\ -2 & 1 \end{array} \right] \left[ \begin{array}{ccc} 3 & 4 \\ 2 & 3 \end{array} \right] = \left[ \begin{array}{cc} 11 & 14 \\ -4 & -5 \end{array} \right].
\]

Let's check basis vectors $W$ are mapped correctly.

Let $v = w_1 = 1w_1 + 0w_2$. So $w = [v]_W = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]$ and $[v]_U = u = (U^{-1}W) w = \left[ \begin{array}{cc} 11 & 14 \\ -4 & -5 \end{array} \right] \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] = \left[ \begin{array}{c} 11 \\ -4 \end{array} \right]$. So $11u_1 - 4u_2 = 11 \left[ \begin{array}{c} 1 \\ 2 \end{array} \right] - 4 \left[ \begin{array}{c} 2 \\ 5 \end{array} \right] = \left[ \begin{array}{c} 3 \\ 2 \end{array} \right] = [v]_E$, as required.

Let $v = w_2 = 0w_1 + 1w_2$. So $w = [v]_W = \left[ \begin{array}{c} 0 \\ 1 \end{array} \right]$ and $[v]_U = u = (U^{-1}W) w = \left[ \begin{array}{cc} 11 & 14 \\ -4 & -5 \end{array} \right] \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] = \left[ \begin{array}{c} 14 \\ -5 \end{array} \right]$. So $14u_1 - 5u_2 = 14 \left[ \begin{array}{c} 1 \\ 2 \end{array} \right] - 5 \left[ \begin{array}{c} 2 \\ 5 \end{array} \right] = \left[ \begin{array}{c} 4 \\ 3 \end{array} \right] = [v]_E$, as required.

The calculator made things easy in the first item of the solution since it can compute a transition matrix automatically. You simply type $U^{-1}W$ to get the final result. Here's how to do it by hand.

Transform the augmented matrix $[U | W]$ into its reduced row echelon form, $[I | U^{-1}W]$, the right partition of which is $U^{-1}W$, the transition matrix. Here it is via elementary row operations.

\[
\left[ \begin{array}{ccc} 1 & 2 & 3 & 4 \\ 2 & 5 & 2 & 3 \end{array} \right] \rightarrow \left[ \begin{array}{ccc} 1 & 2 & 3 & 4 \\ 0 & 1 & -4 & -5 \end{array} \right] \rightarrow \left[ \begin{array}{ccc} 1 & 0 & 11 & 14 \\ 0 & 1 & -4 & -5 \end{array} \right]
\]

159/6

Let $W = \{w_1, w_2, w_3\}$ and $U = \{u_1, u_2, u_3\}$ be bases for $\mathbb{R}^3$ specified by columns of $W$ and $U$ below, respectively.

\[
W = \left[ \begin{array}{ccc} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{array} \right] \quad U = \left[ \begin{array}{ccc} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{array} \right]
\]

(a) Find the transition matrix from $W$ to $U$.

(b) If $x = 2w_1 + 3w_2 - 4w_3$, determine the coordinates of $x$ with respect to $U$.

(c) As a check, compute $[x]_E$ via $Ww$ and $Uu$. 

Solution

(a) We have \( \mathbf{Uu} = \mathbf{Ww} \), whence \( \mathbf{u} = (\mathbf{U}^{-1} \mathbf{W}) \mathbf{w} \). So the desired transition matrix is

\[
\mathbf{U}^{-1} \mathbf{W} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.
\]

(b) Via the transition matrix,

\[
[\mathbf{x}] = \mathbf{u} = (\mathbf{U}^{-1} \mathbf{W}) \mathbf{w} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ -2 \end{bmatrix}.
\]

(c) We verify that

\[
\mathbf{Ww} = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 9 \end{bmatrix} = [\mathbf{x}]_{E},
\]

\[
\mathbf{Uu} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 9 \end{bmatrix} = [\mathbf{x}]_{E}.
\]

159/8

Given \( \mathbf{w}_1 = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \), \( \mathbf{w}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \), and \( \mathbf{T} = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} \), find vectors \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) so that \( \mathbf{T} \) will be the transition matrix from \( \mathbf{W} = \{\mathbf{w}_1, \mathbf{w}_2\} \) to \( \mathbf{U} = \{\mathbf{u}_1, \mathbf{u}_2\} \).

Solution

We have \( \mathbf{Uu} = \mathbf{Ww} \). Thus \( \mathbf{u} = (\mathbf{U}^{-1} \mathbf{W}) \mathbf{w} \). The desired transition matrix is \( \mathbf{T} = \mathbf{U}^{-1} \mathbf{W} \), whence

\[
\mathbf{UT} = \mathbf{W},
\]

\[
\mathbf{U} = \mathbf{WT}^{-1}
\]

\[
= \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}^{-1}
\]

\[
= \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix}.
\]

Thus \( \mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \) and \( \mathbf{u}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \).