Math 311: Topics in Applied Math 1
3: Vector Spaces
3.6: Row Space and Column Space

Summary

- Let $A \in \mathbb{R}^{m \times n}$ be a real $m \times n$ matrix. The subspace of $\mathbb{R}^{1 \times n}$ spanned by the $m$ row vectors of $A$ is called the row space of $A$. The subspace of $\mathbb{R}^{m \times n}$ spanned by the $n$ column vectors of $A$ is called the column space of $A$.

- Row-equivalent matrices have the same row space.

- The rank of a matrix, denoted \( \text{rank}(A) \), is the dimension of the row space of $A$.

- Consistency Theorem for Linear Systems
  A linear system $Ax = b$ is consistent [has solution(s)] if and only if $b$ is in the column space of $A$.

- For an $m \times n$ matrix $A$, the linear system $Ax = b$ is consistent for every $b \in \mathbb{R}^m$ if and only if the column vectors of $A$ span $\mathbb{R}^m$. The system has at most one solution for every $b \in \mathbb{R}^m$ if and only if the column vectors of $A$ are linearly independent.

- A square $n \times n$ matrix $A$ is nonsingular if and only if the column vectors of $A$ form a basis for $\mathbb{R}^n$.

- The Rank-Nullity Theorem
  The nullity of a matrix is the dimension of its null space. For an $m \times n$ matrix $A$, the rank of $A$ plus the nullity of $A$ equals $n$, the number of columns of $A$.

- If $A$ is an $m \times n$ matrix, the dimension of the row space of $A$ is equal to the dimension of the column space of $A$.

- To find a basis for the column space of a matrix $A$, proceed as follows.
  - Obtain its reduced row echelon form $U$ (say via \texttt{rref} on your calculator).
  - Determine the columns of $U$ that correspond to leading 1’s.
  - Then the corresponding columns of $A$ form a basis for the column space of $A$.

Examples

Many of the exercises are facilitated by your calculator’s \texttt{rref} command. Accordingly, please study the calculator video carefully.

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Let $A$ be an $m \times n$ matrix. Prove $\text{rank}(A) \leq \min(m,n)$.

Solution

- The rank of $A$ is the dimension of the row space of $A$. Thus $\text{rank}(A) \leq m$, the number of rows of $A$.
- The dimension of the column space of $A$ is $\leq n$, the number of columns of $A$.
- The dimension of the row space of $A$, $\text{rank}(A)$, is equal to the dimension of the column space of $A$. Thus $\text{rank}(A) \leq n$.
- Accordingly, $\text{rank}(A) \leq \min(m,n)$.

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Let $A$ be a $4 \times 5$ matrix and let $U$ be the reduced row echelon form of $A$. If

$$
A_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

then

(a) find a basis for $N(A)$, the null space of $A$.
(b) given that $x_0$ is a solution of $Ax = b$, where

$$
b = \begin{bmatrix} 0 \\ 5 \\ 3 \\ 4 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \end{bmatrix}
$$

(i) find all solutions to the system.
(ii) determine the remaining column vectors of $A$.

Solution

(a) From $U$, vectors in the null space of $A$ satisfy

$$
\begin{bmatrix} t - 2s \\ 2t - 3s \\ s \\ -5t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 0 \\ -5 \\ 1 \end{bmatrix} = sz_1 + tz_2, \quad s, t \in \mathbb{R},
$$

where $\{z_1, z_2\}$ form a basis for the null space of $A$.
(b) (i) If $y$ is a solution of $Ax = b$, then

$$
A(y - x_0) = Ay - Ax_0 = b - b = 0,
$$
whence $y - x_0 \in N(A)$. Thus $y = x_0 + sz_1 + tz_2$ for some $s, t \in \mathbb{R}$. So solutions of $Ax = b$ have the form

$$y = x_0 + sz_1 + tz_2, \quad s, t \in \mathbb{R}.$$ 

(ii) We can determine the 3rd column of $A$ from $U$, as follows.

$$a_3 = 2a_1 + 3a_2 = \begin{bmatrix} 1 \\ 8 \\ 3 \\ -1 \end{bmatrix}$$

If we let

$$a_4 = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix}$$

then

$$a_5 = 5a_4 - 1a_1 - 2a_2 = \begin{bmatrix} 5a_{14} \\ 5a_{24} - 5 \\ 5a_{34} - 3 \\ 5a_{44} \end{bmatrix}.$$ 

Form $A$ by joining columns (via augment on your calculator). Then $Ax_0 = b$ yields

$$2a_{14} + 4 = 0$$
$$2a_{24} + 7 = 5$$
$$2a_{34} - 3 = 3$$
$$2a_{44} - 4 = 4$$

whence

$$a_4 = \begin{bmatrix} -2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \quad a_5 = \begin{bmatrix} -10 \\ -10 \\ 12 \\ 20 \end{bmatrix}.$$ 

Therefore,

$$A = \begin{bmatrix} 2 & -1 & 1 & -2 & -10 \\ 1 & 2 & 8 & -1 & -10 \\ -3 & 3 & 3 & 3 & 12 \\ -2 & 1 & -1 & 4 & 20 \end{bmatrix}.$$ 

You can now verify that the reduced row echelon form of $A$ is indeed $U$ and that $Ax_0 = b$. Voila!