Math 311: Topics in Applied Math 1
5: Orthogonality
5.1: The Scalar Product in $\mathbb{R}^n$

Summary
Let $x, y \in \mathbb{R}^n$.

- The scalar product of $x$ and $y$ is $x^T y = \sum_{k=1}^{n} x_k y_k$.
- The Euclidean length of $x$ is $\|x\| = \sqrt{x^T x}$.
- The distance between $x$ and $y$ is $\|y - x\|$.
- The angle $\theta$ between two vectors $x$ and $y$ satisfies $x^T y = \|x\| \|y\| \cos \theta$.
- For unit vectors $u$ and $v$, we have $\cos \theta = u^T v$.
- Cauchy-Schwarz Inequality: $|x^T y| \leq \|x\| \|y\|$.
- Vectors $x$ and $y$ are orthogonal if $x^T y = 0$. In this case, we write $x \perp y$.
- For nonzero vectors $x$ and $y$, the scalar projection $\alpha$, vector projection $p$, and orthogonal projection $q$ of $x$ onto $y$ are respectively given by

$$\alpha = \text{comp}_y x = \frac{x^T y}{\|y\|} \quad \text{proj}_y x = \alpha \left( \frac{1}{\|y\|} y \right) = \left( \frac{x^T y}{y^T y} \right) y$$

$$q = \text{orth}_y x = x - \text{proj}_y x$$

Let $P_1$ and $P_2$ be points in 3-dimensional real space. We identify them with their position vectors $P_1$ and $P_2$ in $\mathbb{R}^3$. $P_1P_2$ denotes the vector $P_2 - P_1 \in \mathbb{R}^3$.

- Given a nonzero vector $n \in \mathbb{R}^3$ and a fixed point $P_0$ in 3-dimensional space, the set of points $P$ in space such that $n \perp P_0P$ is a plane. Here

$$n^T (P - P_0) = 0$$

or $n^T P = n^T P_0$

We say the vector $n$ and the plane are normal to each other; $n$ is called a normal vector.

Examples
Also watch the calculator video for more examples.

224/4

Let $x$ and $y$ be linearly independent vectors in $\mathbb{R}^2$. Given $\|x\| = 2$ and $\|y\| = 3$, what can we conclude about the possible values of $|x^T y|$?

Solution
Since $x$ and $y$ be linearly independent, neither vector is a multiple of the other. Hence the angle $\theta$ between them is not a multiple of $\pi$. Hence $|\cos \theta| < 1$. Therefore,

$$|x^T y| = \|x\| \|y\| \cos \theta = \|x\| \|y\| |\cos \theta| < \|x\| \|y\| < (2)(3)$$

or $|x^T y| < 6$.

224/9

Find an equation of the plane that passes through the points $P_1 (2, 3, 1)$, $P_2 (5, 4, 3)$, and $P_3 (3, 4, 4)$.

Solution
- Let $v = P_1P_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and $w = P_1P_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

Back in Calc 3 we'd compute our normal vector as the cross product $n = v \times w$. For grins, let's alternatively directly obtain a vector that is orthogonal to both $v$ and $w$.

- Let $n = \begin{bmatrix} a \\ b \\ c \end{bmatrix}^T$. Then $n^T v = 0$ and $n^T w = 0$. This gives a linear system in the variables $a, b, c$, whose augmented system matrix and reduced row echelon form are

$$\begin{bmatrix} 3 & 1 & 2 & 0 \\ 1 & 1 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 7/2 & 0 \end{bmatrix}.$$

- Thus $a = \frac{1}{2}c$ and $b = -\frac{7}{2}c$. So $n = \begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix}^T$ is a normal vector. Hence an equation of the plane is

$$n^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = n^T P_1$$

$$x - 7y + 2z = -17.$$ 

225/12+

Prove that if $x, y, z \in \mathbb{R}^n$, then

(a) $x^T x \geq 0$  (b) $x^T y = y^T x$  (c) $x^T (y + z) = x^T y + x^T z$

Solution
(a) We have $x^T x = \sum_{k=1}^{n} x_k^2 \geq 0$.

(b) Since $a = x^T y$ is a scalar ($1 \times 1$ matrix), we have

$$x^T y = a = a^T = (x^T y)^T = y^T x$$

whence $x^T y = y^T x$.

(c) We have $x^T (y + z) = x^T y + x^T z$ since left matrix multiplication distributes over matrix addition.