Math 311: Topics in Applied Math 1
5: Orthogonality
5.2: Orthogonal Subspaces

Summary

- Subspaces X and Y of \( \mathbb{R}^n \) are orthogonal if \( x^T y = 0 \)
  for every vector \( x \in X \) and \( y \in Y \). We write \( X \perp Y \) to signify this. If \( X \perp Y \), then \( X \cap Y = \{0\} \); that is, their intersection consists solely of the zero vector.

- Let \( Y \) be a subspace of \( \mathbb{R}^n \). The orthogonal complement of \( Y \) is
  \[ Y^\perp = \{ x \in \mathbb{R}^n : x^T y = 0 \text{ for every } y \in Y \} , \]
  the set of all vectors in \( \mathbb{R}^n \) that are orthogonal to every vector in \( Y \). Note \( Y^\perp \) is also a subspace of \( \mathbb{R}^n \).

- Regard an \( m \times n \) real matrix \( A \) as representing a linear transformation \( L : \mathbb{R}^n \rightarrow \mathbb{R}^m \). Denote the range of \( A \) by \( R(A) \).
  \[ R(A) = \{ b \in \mathbb{R}^m : b = Ax \text{ for some } x \in \mathbb{R}^n \} = \text{column space of } A , \text{ a subspace of } \mathbb{R}^m \]

- Similarly, the range of \( A^T \) is a subspace of \( \mathbb{R}^n \).
  \[ R(A^T) = \{ y \in \mathbb{R}^n : y = A^T x \text{ for some } x \in \mathbb{R}^m \} = \text{column space of } A^T , \text{ a subspace of } \mathbb{R}^n \]

- Fundamental Subspaces Theorem
  If \( A \) is an \( m \times n \) real matrix, then
  \[ N(A) = R(A^T)^\perp \text{ and } N(A^T) = R(A)^\perp . \]

- If \( S \) is a subspace of \( \mathbb{R}^n \), then \( \dim S + \dim S^\perp = n \). Moreover, if \( \{ x_i \}_{i=1}^r \) is a basis for \( S \) and \( \{ x_j \}_{j=r+1}^n \) is a basis for \( S^\perp \), then \( \{ x_i \}_{i=1}^r \) is a basis for \( \mathbb{R}^n \).

- If \( U \) and \( V \) are subspaces of a vector space \( W \) and each \( w \in W \) can be written as a sum \( w = u + v \) uniquely, then we say that \( W \) is a direct sum of \( U \) and \( V \), written \( W = U \oplus V \).

- If \( S \) is a subspace of \( \mathbb{R}^n \), then \( \mathbb{R}^n = S \oplus S^\perp \).

- If \( S \) is a subspace of \( \mathbb{R}^n \), then \( (S^\perp)^\perp = S \).

- If \( A \) is an \( m \times n \) real matrix and \( b \in \mathbb{R}^m \), then either \( Ax = b \) for some \( b \in \mathbb{R}^n \) or for some \( y \in \mathbb{R}^m \) we have \( A^Ty = 0 \) with \( y^Tb \neq 0 \).

Examples

233/1c

For the matrix \( A = \begin{bmatrix} 4 & -2 \\ 1 & 3 \\ 2 & 1 \\ 3 & 4 \end{bmatrix} \), determine a basis for each of the subspaces \( R(A^T), N(A), R(A), \) and \( N(A^T) \).

Solution

- The reduced row echelon form of \( A \) is
  \[ U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} . \]

- Now \( [ 1 \ 0 ] \) and \( [ 0 \ 1 ] \) form a basis for the row space of \( U \) as well as \( A \) (since these matrices are row equivalent). Accordingly, the vectors \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) form a basis for \( R(A^T) \), the column space of \( A^T \).

- If \( x \in N(A) \), it follows from the reduced row echelon form of \( A \) that \( x_1 = 0 \) and \( x_2 = 0 \). So \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \) forms a basis for \( N(A) = \{0\} \).

- Since the column vectors in the reduced row echelon form of \( A \) are linearly independent, the corresponding column vectors of \( A \),
  \[ \begin{bmatrix} 4 \\ 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 3 \\ 1 \\ 4 \end{bmatrix} , \]
  form a basis for \( R(A) \).

- The row reduced echelon form of \( A^T \) is
  \[ \begin{bmatrix} 1 & 0 & 5/3 & 5/3 \\ 0 & 1 & 4/7 & 11/7 \end{bmatrix} . \]

- The null space of \( A^T \) consists of vectors of the form
  \[ \begin{bmatrix} -5 & 5 & -8 & 14s \\ -5 \tau - 22t & 8s - 22t & 14s & 14 \end{bmatrix} = s \begin{bmatrix} -5 \\ -8 \end{bmatrix} + t \begin{bmatrix} -5 \\ -22 \end{bmatrix} \]
  with \( s, t \in \mathbb{R} \). Thus \[ \begin{bmatrix} -5 \\ -8 \\ 14 \end{bmatrix} \] and \[ \begin{bmatrix} -5 \\ -22 \end{bmatrix} \] form a basis for \( N(A^T) \).
Is it possible for a matrix $A$ to have $v = \begin{bmatrix} 3 & 1 & 2 \end{bmatrix}$ in its row space and $w = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}^T$ in its null space?

**Solution**

Assume it is possible. Since $v$ is in the row space of $A$, we have $y = v^T \in R(A^T)$. Also, $w \in N(A) = R(A^T)^\perp$ implies that $w$ is orthogonal to every vector in $R(A^T)$. Hence $w$ is orthogonal to $y$; so $w^T y = 0$. However,

$$w^T y = 6 + 1 + 2 = 9,$$

a contradiction since $0 \neq 9$. Therefore the assumption is false and it is not possible for such a matrix to exist.

If $A$ is an $m \times n$ matrix of rank $r$, what are the dimensions of $N(A)$ and $N(A^T)$? Explain.

**Solution**

- By the Rank-Nullity Theorem of Section 3.6, the rank of $A$ plus the nullity of $A$ (the dimension of the null space of $A$) equals $n$, the number of columns of $A$. Since the rank of $A$ is $r$, the dimension of $N(A)$ is $n - r$.

- The rank of $A$ is the dimension of the row space of $A$. This is also the dimension of the column space of $A^T$. So both of these equal $r$. Another application of the Rank-Nullity Theorem tells us that the dimension of $N(A^T)$ is $n - r$.

- So $N(A)$ and $N(A^T)$ have the same dimension; namely, $n - r$.

Let $x$ and $y$ be linearly independent vectors in $\mathbb{R}^n$ and let $S = \text{Span}(x, y)$. We can use $x$ and $y$ to define a matrix $A$ by setting $A = xy^T + yx^T$.

(a) Show that $A$ is symmetric.

(b) Show that $N(A) = S^\perp$.

(c) Show that the rank of $A$ is 2.

Thus $A^T = A$, whence $A$ is symmetric.

(b) Let’s show that $S = R(A)$, the column space of $A$. Then we’ll conclude $N(A) = S^\perp$.

- First we show that $R(A) \subset S$. Let $c \in R(A)$. Then

$$Ac = (xy^T + yx^T)c = x(y^T c) + y(x^T c) = \alpha x + \beta y \in S.$$ 

Hence $R(A) \subset S$.

- Now we show that $S \subset R(A)$. Recall Corollary 2.5 on page 231 of this section and the fact that $A$ is symmetric from part (a). Therefore,

- either there is a vector $v \in \mathbb{R}^n$ such that $Av = x$
- or there is a vector $w \in \mathbb{R}^n$ such that $Aw = 0$

and $y = w^T x \neq 0$ whence $c_2 = y^T = x^T w \neq 0$.

If the second case is true, then

$$0 = Aw = (xy^T + yx^T)w = x(y^T w) + y(x^T w) = c_1 x + c_2 y.$$ 

Since $c_2 \neq 0$, there exist scalars $c_1$ and $c_2$ not both zero such that $c_1 x + c_2 y = 0$, contradicting the fact that $x$ and $y$ are linearly independent from the statement of the problem. Hence the first case must be true, in which case $x$ is in the column space of $A$.

Similarly, $y$ is in the column space of $A$. Therefore any linear combination $\alpha x + \beta y$ is in $R(A)$, the column space of $A$, since $R(A)$ is a subspace of $\mathbb{R}^n$.

Accordingly, $S \subset R(A)$.

- Thus $S = R(A)$. By the Fundamental Subspaces Theorem and the fact that $A$ is symmetric, we have

$$N(A) = R(A^T)^\perp = R(A)^\perp = S^\perp;$$

that is, $N(A) = S^\perp$.

(c) Since $x$ and $y$ are linearly independent, $\dim S = 2$. Thus $\dim S + \dim S^\perp = n$ and $N(A) = S^\perp$ imply $\dim N(A) = n - 2$. By the Rank-Nullity Theorem, the rank of $A$ is $n - (n - 2) = 2$. 

233/6