Math 311: Topics in Applied Math 1
5: Orthogonality
5.4: Inner Product Spaces

Summary

Here $V$ is an inner product space, as defined below.

- An inner product on a vector space $V$ is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that the following conditions are satisfied for vectors $x, y \in V$.
  - 1. $\langle x, x \rangle \geq 0$, with equality iff $x = 0$.
  - 2. $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in V$.
  - 3. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, for all $x, y, z \in V$ and all $\alpha, \beta \in \mathbb{R}$.

- A vector space $V$ for which an inner product is defined is called an inner product space.

- The length (or standard norm) of $v \in V$ is $\|v\| = \sqrt{\langle v, v \rangle}$.

- Vectors $u, v \in V$ are orthogonal if $\langle u, v \rangle = 0$.

- The Pythagorean Law: If $u$ and $v$ are orthogonal, then $\|u + v\|^2 = \|u\|^2 + \|v\|^2$.

- For vectors $u$ and $v \neq 0$ in an inner product space $V$, the scalar projection $\alpha$, vector projection $p$, and orthogonal projection $q$ of $u$ onto $v$ are defined by
  
  $\alpha = \frac{\langle u, v \rangle}{\|v\|^2}$
  
  $p = \alpha \left( \frac{1}{\|v\|^2} \right) v = \frac{\langle u, v \rangle}{\langle v, v \rangle} v$
  
  $q = u - p$.

Observe that $u - p$ and $p$ are orthogonal and $u = p$ iff $u = \beta v$ for some scalar $\beta$. (See page 248.)

- Cauchy-Schwarz Inequality: For $u, v$ in an inner product space $V$, we have $|\langle u, v \rangle| \leq \|u\| \|v\|$. Equality holds iff $u$ and $v$ are linearly dependent.

- The angle $\theta$ between two nonzero vectors $u$ and $v$ in an inner product space is $\theta \in [0, \pi]$ such that
  
  $\cos \theta = \frac{\langle u, v \rangle}{\|u\| \|v\|}$.

- A vector space $V$ together with a function $\| \cdot \| : V \to \mathbb{R}$, is called a normed linear space provided these conditions are satisfied for all $v, w \in V$. (The function $\| \cdot \|$ is called a norm.)
  
  - 1. $\|v\| \geq 0$ with equality iff $v = 0$.
  - 2. $\|\alpha v\| = |\alpha| \|v\|$ for any scalar $\alpha \in \mathbb{R}$.
  - 3. $\|v + w\| \leq \|v\| + \|w\|$ (Triangle Inequality)

- THEOREM: If $V$ is an inner product space, then $\|v\| = \sqrt{\langle v, v \rangle}$ for all $v \in V$ defines a norm on $V$.

- For vectors $x$ and $y$ in a normed linear space, the distance between $x$ and $y$ is $\|x - y\|$.

Textbook and Hand Examples

Inner Product Spaces

These examples are mentioned in your textbook.

1. $V = \mathbb{R}^n$ with $\langle x, y \rangle = x^T y = \sum x_k y_k$. This standard inner product is the scalar product from Section 5.1.

2. $V = \mathbb{R}^n$ with $\langle x, y \rangle = \sum x_k y_k w_k$, where $w$ is a vector with positive elements, called weights.

3. $V = C[a, b]$, the vector space of continuous functions on $[a, b]$, with $\langle f, g \rangle = \int_a^b f(x) g(x) \, dx$.

4. $V = C[a, b]$ with $\langle f, g \rangle = \int_a^b f(x) g(x) w(x) \, dx$, where $w \in C[a, b]$, a weight function, takes on positive values.

5. $V = P_n$, the set of polynomials of degree $\leq n$, with $\langle p, q \rangle = \sum_{k=1}^n x_k^p (q(x_k))$ for distinct $\{x_k\}_{k=1}^n \subset \mathbb{R}$.

6. $V = P_n$ with $\langle p, q \rangle = \sum_{k=1}^n x_k^p (q(x_k)) w(x_k)$ for distinct $\{x_k\}_{k=1}^n \subset \mathbb{R}$ and where $w \in C(\mathbb{R})$ is positive.

7. $V = \mathbb{R}^{m \times n}$, the set of real $m \times n$ matrices, with $\langle A, B \rangle = \sum_{i=1}^m \sum_{j=1}^n a_{ij} b_{ij}$

Norms on $\mathbb{R}^n$

For brevity, let $\Sigma$ signify $\sum_{k=1}^n$.

- $p$-norm: $\|x\|_p = (\Sigma |x_k|^p)^{1/p}$, where $p \geq 1$

- 1-norm (p-norm with $p = 1$): $\|x\|_1 = \Sigma |x_k|$

- 2-norm (p-norm with $p = 2$): $\|x\|_2 = (\Sigma |x_k|^2)^{1/2}$

- $\infty$-norm (p-norm as $p \to \infty$): $\|x\|_\infty = \max_{1 \leq k \leq n} |x_k|$

Norms on $\mathbb{R}^{m \times n}$

- Frobenius norm:
  
  $\|A\|_F = \sqrt{\langle A, A \rangle} = \left( \sum_{i=1}^m \sum_{j=1}^n a_{ij}^2 \right)^{1/2}$

- Other matrix norms are discussed in Chapter 7: Numerical Linear Algebra. We won’t cover that chapter this term, but you may encounter the ideas in a future course.
In \( \mathbb{R}^n \) with the standard inner product \( \langle x, y \rangle = x^T y \), derive a formula for the distance between two vectors.

**Solution**

With norm \( \|v\| = \sqrt{\langle v, v \rangle} \), the distance is \( \|x - y\| \).

\[
\|x - y\| = \sqrt{\langle x - y, x - y \rangle} = \sqrt{(x^T - y^T)(x - y)} = \sqrt{x^T x - x^T y - y^T x + y^T y} = \sqrt{x^T x - 2x^T y + y^T y + 2\langle x^T y \rangle} = \sqrt{\|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \cos \theta} = \sqrt{\|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \cos \theta}.
\]

where the integer summation index \( k \) ranges from 1 to \( n \). In step \( * \), we used the fact that the transpose of a real scalar \((1 \times 1 \text{ real matrix})\) is itself.

**Computational Examples**

I highly recommend doing computations on your calculator. It helps immensely with these types of problems from this section. That said, here are a few examples to show you the flow!

**252/19**

For \( x = [1 \ 1 \ 1 \ 1 \ 1]^T \) and \( y = [8 \ 2 \ 2 \ 0 \ 0]^T \) in vector space \( \mathbb{R}^4 \) with standard inner product and norm:

(a) Determine the angle \( \theta \) between \( x \) and \( y \).
(b) Find the vector projection \( p \) of \( x \) onto \( y \).
(c) Verify that \( q = x - p \) is orthogonal to \( p \).
(d) Compute \( \|x - p\|, \|p\|, \|x\| \). Verify that the Pythagorean Law is satisfied.

**Solution**

(a) \( \cos \theta = \frac{x^T y}{\|x\|\|y\|} = \frac{12}{2 \sqrt{6}} = \frac{\sqrt{6}}{2} \), so \( \theta = \frac{\pi}{4} \).

(b) The vector projection is \( p = \frac{x^T y}{y^T y} y = \frac{12}{72} y = [\frac{4}{3} \ \frac{1}{3} \ \frac{1}{3} \ 0]^T \).

(c) The orthogonal projection is \( q = x - p = [-\frac{1}{3} \ \frac{2}{3} \ \frac{2}{3} \ 1]^T \).

Since \( p^T q = -\frac{4}{3} + \frac{2}{3} + \frac{2}{3} + 0 = 0 \), \( p \) and \( q \) are orthogonal.

(d) Since \( \|x - p\|^2 + \|p\|^2 = 2 + 2 = 4 = \|x\|^2 \), the Pythagorean Law is satisfied.

**251/2**

For the vector space \( C[0,1] \) with inner product \( \langle f, g \rangle = \int_0^1 f(x) g(x) \, dx \) and norm \( \|h\| = \sqrt{\langle h, h \rangle} \), consider vectors \( f = 1 \) and \( g = x \).

(a) Determine the angle \( \theta \) between \( f \) and \( g \).
(b) Find the vector projection \( p \) of \( f \) onto \( g \).
(c) Verify that \( q = f - p \) is orthogonal to \( p \).
(d) Compute \( \|f - p\|, \|p\|, \|f\| \). Verify that the Pythagorean Law is satisfied.
Solution

Conceptually, this is identical to the preceding problem. This is the beauty of abstraction, an acquired taste. Look at the calculator video, where the abstraction is hidden.

(a) We observe
\[
\cos \theta = \frac{\langle f, g \rangle}{\|f\| \|g\|} = \frac{\int_0^1 x \, dx}{\sqrt{\int_0^1 1 \, dx \sqrt{\int_0^1 x \, dx}}} = \frac{1/2}{(1)(1/\sqrt{3})} = \frac{\sqrt{3}}{2},
\]
so \(\theta = \frac{\pi}{6}\).

(b) The vector projection is
\[
p = \frac{\langle f, g \rangle}{\langle g, g \rangle} g = \frac{\langle f, g \rangle}{\|g\|^2} g = \frac{1/2}{1/2} x = \frac{3}{2} x.
\]

(c) The orthogonal projection is
\[
q = f - p = 1 - \frac{3}{2} x.
\]

Since \(\langle p, q \rangle = \int_0^1 1/2 x \, dx = 0\), we see that \(p\) and \(q\) are orthogonal.

(d) Note that
\[
\|f - p\|^2 + \|p\|^2 = \langle f - p, f - p \rangle + \langle p, p \rangle
= \int_0^1 \left(1 - \frac{3}{2} x\right)^2 \, dx + \int_0^1 \left(\frac{3}{2} x\right)^2 \, dx
= \frac{1}{4} + \frac{3}{4} = 1 = \int_0^1 1/2 \, dx
= \langle 1, 1 \rangle = \langle f, f \rangle = \|f\|^2,
\]
so \(\|f - p\|^2 + \|p\|^2 = \|f\|^2\). The Pythagorean Law holds.

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Given \(x = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}\) and \(y = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}\) in \(\mathbb{R}^3\), compute \(\|x - y\|_1\), \(\|x - y\|_2\), and \(\|x - y\|_\infty\).

Solution

All three of these norms for \(\mathbb{R}^n\) are built into the calculator operating system, as \texttt{colNorm}, \texttt{norm}, and \texttt{rowNorm}, respectively. Now \(v = x - y = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}\). So
\[
\begin{align*}
\|x - y\|_1 &= \sum |v_k| = 2 + 1 + 2 = 5 \\
\|x - y\|_2 &= \left(\sum |v_k|^2\right)^{1/2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3 \\
\|x - y\|_\infty &= \max |v_k| = 2.
\end{align*}
\]