Math 311: Topics in Applied Math 1
5: Orthogonality
5.5: Orthonormal Sets

Summary

Throughout the summary, V is an inner product space. For brevity, \( \Sigma \) stands for summation over an index set.

- Let \( \{v_i\}_{i=1}^n \in V \) be nonzero vectors. If \( \langle v_i, v_j \rangle = 0 \) for \( i \neq j \), these form an orthogonal set of vectors.
- An orthogonal set has linearly independent vectors.
- We call an orthogonal set of unit vectors \( \{u_1, \ldots, u_k\} \) an orthonormal set. Note that
  \[
  \langle u_i, u_j \rangle = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}
  \]
- Orthonormal vectors \( \{u_1, \ldots, u_k\} \) form an orthonormal basis for \( V = \text{span} (u_1, \ldots, u_k) \).
- Let \( \{u_1, \ldots, u_n\} \) be an orthonormal basis for \( V \).
  - If \( v = \Sigma a_i u_i \), then \( a_i = \langle v, u_i \rangle \), \( i = 1, \ldots, n \).
  - If \( w = \Sigma b_i u_i \), then \( \langle v, w \rangle = \Sigma a_i b_i \).
  - \( \|v\|^2 = \langle v, v \rangle = \Sigma a_i^2 \) (Parseval’s formula)
- An \( n \times n \) orthogonal matrix \( Q \) is one whose column vectors form an orthonormal set in \( \mathbb{R}^n \).
  - Its columns form an orthonormal basis for \( \mathbb{R}^n \).
  - \( Q^T Q = I \)
  - \( Q^T = Q^{-1} \)
  - \( \langle Qx, Qy \rangle = \langle x, y \rangle \): Inner products are preserved under multiplication by \( Q \).
  - \( \|Qx\|_2 = \|x\|_2 \): Lengths (in the 2-norm) are preserved under multiplication by \( Q \).
- An \( n \times n \) permutation matrix \( P \) is one obtained from the identity matrix by reordering its columns. It is an orthogonal matrix. Let \( A \) be an \( m \times n \) matrix and \( B \) an \( n \times r \) matrix.
  - \( PB \) reorders rows of \( B \).
  - \( AP \) reorders columns of \( A \).
  - Both of these reorderings reflect that of \( P \) from \( I \).
- If the column vectors of an \( m \times n \) matrix \( A \) form an orthonormal set in \( \mathbb{R}^n \), then \( A^T A = I \), and hence \( \hat{x} = A^T b \) is the unique solution of the least squares problem \( Ax = b \).
- Let \( S \) be a subspace of \( V \) with orthonormal basis \( \{u_1, \ldots, u_n\} \) and let \( x \in V \). If \( p = \Sigma c_i u_i \) where \( c_i = \langle x, u_i \rangle \), then \( p - x \in S^\perp \). Moreover, \( p \) is the element of \( S \) that is closest to \( x \). In other words, \( \|y - x\| \geq \|p - x\| \) for any \( y \neq p \) in \( S \). The vector \( p \) is the projection of \( x \) onto \( S \).
- Let \( S \neq \{0\} \) be a subspace of \( \mathbb{R}^m \) and let \( b \in \mathbb{R}^m \). Let \( U = [u_1, \ldots, u_k] \) be the \( m \times k \) matrix whose columns form an orthonormal basis for \( S \). Then the projection \( p \) of \( b \) onto \( S \) is \( p = UU^T b \). The matrix \( UU^T \) is the projection matrix corresponding to \( S \).

You may find material regarding approximation interesting. In your textbook, see pages 262–269 of Section 5.5 as well as Section 5.7 for details.

Examples

It is much easier to work with an orthonormal basis than with an ordinary basis!

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Which of the following sets form an orthonormal basis for \( \mathbb{R}^2 \)?

(c) \( \{v_1, v_2\} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \)

(d) \( \{w_1, w_2\} = \left\{ \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix} \right\} \)

Solution

(c) The dimension of \( \mathbb{R}^2 \) is 2. While the vectors do form a basis for \( \mathbb{R}^2 \) (they’re linearly independent and there are two of them), they are not unit vectors and hence do not form an orthonormal basis (indeed, \( \|v_1\| = \sqrt{2} \neq 1 \)).

(d) The vectors do form an orthonormal basis for \( \mathbb{R}^2 \): they are orthogonal unit vectors and there are two of them.

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Let \( \{u_1, u_2, u_3\} \) be an orthonormal basis for an inner product space \( V \). Given the vectors \( v = u_1 + 2u_2 + 2u_3 \) and \( w = u_1 + 7u_3 \), determine the following.

(a) \( \langle v, w \rangle \)

(b) \( \|v\| \) and \( \|w\| \)

(c) angle between \( v \) and \( w \)

Solution

(a) We have \( \langle v, w \rangle = (1)(1) + (2)(0) + (2)(7) = 15 \).

(b) We have \( \|v\| = \sqrt{\langle v, v \rangle} = \sqrt{1^2 + 2^2 + 2^2} = 3 \) and \( \|w\| = \sqrt{1^2 + 0^2 + 7^2} = \sqrt{50} = 5\sqrt{2} \).

(c) So \( \theta = \cos^{-1} \left( \frac{\langle v, w \rangle}{\|v\|\|w\|} \right) = \cos^{-1} \left( \frac{15}{5\sqrt{2}} \right) = \frac{\pi}{4} \) radians.
The functions \( \cos x \) and \( \sin x \) form an orthonormal set in \( C[-\pi, \pi] \). Use this fact to compute the inner product 
\[
\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) \, dx
\]
without doing any integration for \( f(x) = 3\cos x + 2\sin x \) and \( g(x) = \cos x - \sin x \).

**Solution**

We have \( \langle f, g \rangle = (3)(1) + (2)(-1) = 1 \), as you may verify via integration on your calculator.

If \( Q \) is an \( n \times n \) orthogonal matrix and \( x \) and \( y \) are nonzero vectors in \( \mathbb{R}^n \), how does the angle between \( Qx \) and \( Qy \) compare with the angle between \( x \) and \( y \)?

**Solution**

Since \( Q \) is an orthogonal matrix, inner products and lengths are preserved under multiplication by \( Q \). Accordingly, the angle between \( Qx \) and \( Qy \) is
\[
\cos^{-1} \left( \frac{\langle Qx, Qy \rangle}{\|Qx\|_2 \|Qy\|_2} \right) = \cos^{-1} \left( \frac{\langle x, y \rangle}{\|x\|_2 \|y\|_2} \right),
\]
the same as the angle between \( x \) and \( y \).

Let \( Q \) be an orthogonal matrix and let \( d = \det(Q) \).
Show that \( |d| = 1 \).

**Solution**

Since \( Q \) is an orthogonal matrix, we have \( Q^T Q = I \).
Recall that the determinants of a square matrix and its transpose are equal. Therefore,
\[
1 = \det(I) = \det(Q^T Q) = \det(Q^T) \det(Q) = d^2
\]
whence \( d^2 = 1 \) and thus \( |d| = 1 \).