

Math 151, Spring 2000
Solutions to
Common Exam 3 Version A

Part 1. Multiple choice problems. Each problem is worth 4 points. Read each question carefully. No calculators are allowed on this part.

1. If $f(x) = \frac{x}{\ln x}$, $f'(e)$ is equal to?

$$f'(x) = \frac{\ln x - x(1/x)}{(\ln x)^2}$$
$$f'(e) = \frac{\ln e - 1}{(\ln e)^2} = \frac{1 - 1}{1} = 0$$

2. What is the solution to the equation $\ln(x+1) = 3 - \ln(x-1)$?

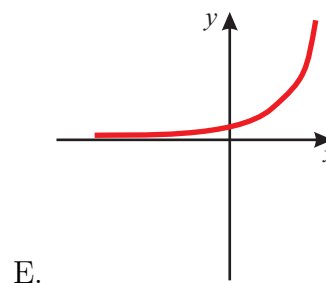
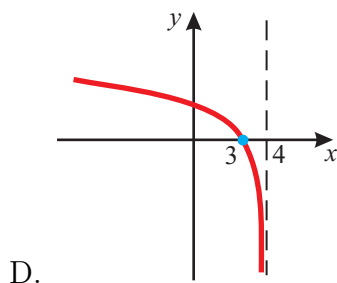
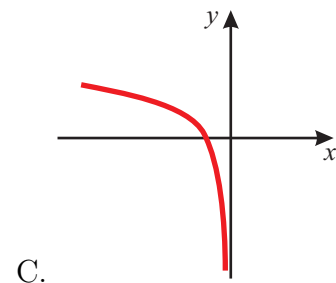
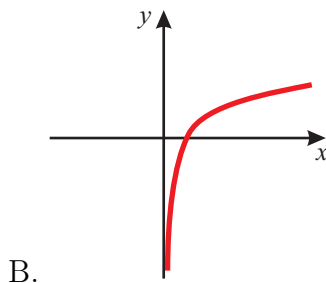
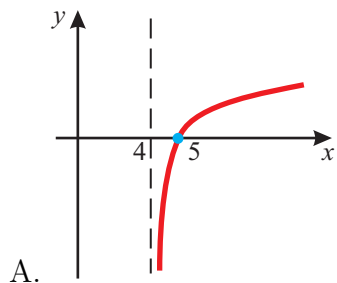
$$\begin{aligned}\ln(x+1) + \ln(x-1) &= 3 \\ \ln[(x+1)(x-1)] &= 3 \\ (x+1)(x-1) &= e^3 \\ x^2 - 1 &= e^3 \\ x &= \pm\sqrt{e^3 + 1}\end{aligned}$$

We discard the negative root since $-\sqrt{e^3 + 1} - 1$ is negative and not in the domain of the natural log function. Thus, the solution is $x = \sqrt{e^3 + 1}$.

3. $\frac{d}{dx}(x^{\sin x})$ equals?

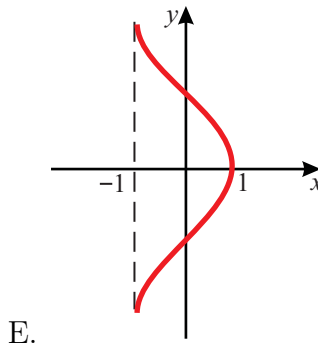
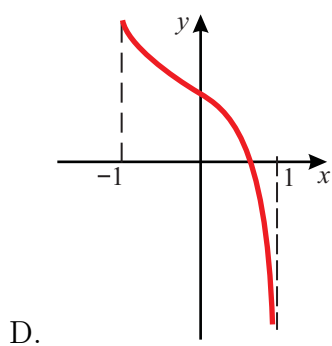
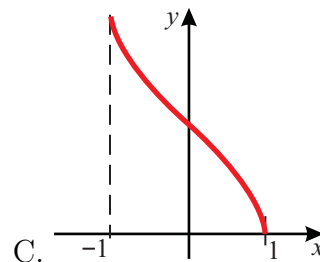
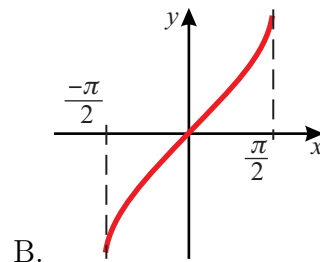
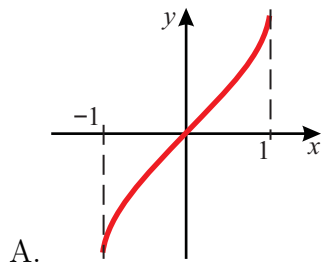
$$\begin{aligned}\frac{d}{dx}(x^{\sin x}) &= \frac{d}{dx}(e^{\ln x \sin x}) = \left[\frac{d}{dx}(\ln x \sin x) \right] e^{\ln x \sin x} \\ &= \left[\frac{\sin x}{x} + (\ln x \cos x) \right] e^{\ln x \sin x}\end{aligned}$$

4. Which of the following is most likely to be the graph of $y = \ln(4 - x)$?



The domain of $\ln(4 - x)$ is the set of x such that $4 - x > 0$ or $x < 4$. Thus, D is the correct graph.

5. Which of the following is most likely the graph of $\cos^{-1} x$?



The domain of $\cos^{-1} x$ is $[-1, 1]$ and its range is $[0, \pi]$. The only plot which has both of these properties is C.

6. $\frac{d}{dx} \sin^{-1}(2x)$ equals?

$$\frac{d}{dx} \sin^{-1}(2x) = \frac{2}{\sqrt{1-4x^2}}$$

7. Evaluate $\lim_{x \rightarrow 1^-} \frac{\pi/2 - \sin^{-1} x}{\sqrt{1-x}}$

$$\begin{aligned} \lim_{x \rightarrow 1^-} \frac{\pi/2 - \sin^{-1} x}{\sqrt{1-x}} &= \lim_{x \rightarrow 1^-} \frac{-\left(\frac{1}{\sqrt{1-x^2}}\right)}{\frac{1}{2}(1-x)^{-1/2}(-1)} = \lim_{x \rightarrow 1^-} 2 \frac{\sqrt{1-x}}{\sqrt{1-x^2}} \\ &= \lim_{x \rightarrow 1^-} 2 \frac{\sqrt{1-x}}{\sqrt{1-x}\sqrt{1+x}} = \lim_{x \rightarrow 1^-} 2 \frac{1}{\sqrt{1+x}} = \frac{2}{\sqrt{2}} = \sqrt{2} \end{aligned}$$

8. Find $\lim_{x \rightarrow 0} (1+x)^{1/x}$.

$$\begin{aligned} \lim_{x \rightarrow 0} (1+x)^{1/x} &= \lim_{x \rightarrow 0} e^{\left(\frac{\ln(1+x)}{x}\right)} = e^{\lim_{x \rightarrow 0} \left(\frac{\ln(1+x)}{x}\right)} \\ &= e^{\lim_{x \rightarrow 0} \frac{1/(1+x)}{1}} = e^1 = e \end{aligned}$$

9. Find $\lim_{x \rightarrow \infty} \tan^{-1}\left(\frac{1}{x^2}\right)$

$$\lim_{x \rightarrow \infty} \tan^{-1}\left(\frac{1}{x^2}\right) = \tan^{-1}(0) = 0$$

10. Let $f(x) = x^2 + 3|x - 1|$. Then the absolute minimum of $f(x)$ on the interval $[0, 2]$

A. is attained at $x = 0$. B. is attained at $x = 2$.

C. is attained at a point $x = c$, where $f'(c) = 0$.

D. is attained at a point $x = c$ where $f'(c)$ does not exist.

E. does not exist.

To handle the absolute value function we examine $f(x)$ on the intervals $[0, 1]$ and $[1, 2]$ separately. On the interval $[0, 1]$, $f(x) = x^2 - 3(x - 1)$. The derivative of this function is $2x - 3$ which is negative on $[0, 1]$. Thus, $f(x)$ is decreasing on the interval $[0, 1]$ and its minimum on $[0, 1]$ occurs at $x = 1$.

On the interval $[1, 2]$, $f(x) = x^2 + 3(x - 1)$. The derivative of this function is $2x + 3$ and it is positive on the interval $[1, 2]$. Thus, $f(x)$ is increasing on the interval $[1, 2]$. Thus, the minimum value of $f(x)$ on the entire interval $[0, 2]$ occurs at the point $c = 1$. A point where the function $x^2 + 3|x - 1|$ does not have a derivative. The correct answer is D.

11. Find a function $g(x)$ such that $g'(x) = \frac{1}{x \ln x}$.

Any antiderivative of $\frac{1}{x \ln x}$ is of the form $\ln(\ln x) + C$.

12. Write the following sum in sigma (\sum) notation.

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \frac{6}{64}$$

$$\text{A. } \sum_{i=2}^7 \frac{i-1}{2^i} \quad \text{B. } \sum_{i=1}^6 \frac{i}{2^i} \quad \text{C. } \sum_{n=1}^6 \frac{n}{n2^n} \quad \text{D. } \sum_{n=1}^6 \frac{n+1}{2^n} \quad \text{E. } \sum_{i=0}^6 \frac{i+1}{2^{i-1}}$$

The denominator is of the form 2^i as i goes from 1 to 6. This eliminates choices C, and E. Since the numerator is the same as the exponent of 2, this eliminates choices A, and D leaving only B as the correct choice.

13. Find $f(x)$ such that $f''(x) = \sqrt{x}$, $f(0) = 1$, $f'(0) = 2$.

$f''(x) = \sqrt{x}$ implies $f'(x) = \frac{2}{3}x^{3/2} + C$. Since $f'(0) = 2$, we know $C = 2$. Next we find an antiderivative of $f'(x)$. $f(x) = \frac{2}{3} \cdot \frac{2}{5}x^{5/2} + 2x + C$. Since $f(0) = 1$, the constant of integration C equals 1. Thus, $f(x) = \frac{4}{15}x^{5/2} + 2x + 1$.

14. Suppose $\int_1^7 f(x) dx = 6$, and $\int_1^4 3f(x) dx = 12$. Find $\int_4^7 f(x) dx$.

$$\begin{aligned} \int_4^7 f(x) dx &= \int_1^7 f(x) dx - \int_1^4 f(x) dx \\ &= \int_1^7 f(x) dx - \frac{1}{3} \int_1^4 3f(x) dx \\ &= 6 - \frac{1}{3}12 = 6 - 4 = 2 \end{aligned}$$

15. To which integral does $\lim_{n \rightarrow \infty} \left(\frac{3}{n} \sum_{i=1}^n \sqrt{1 + \frac{3i}{n}} \right)$ converge?

- A. $\int_0^1 \sqrt{x} dx$ B. $\int_0^3 \sqrt{x} dx$ C. $\int_0^3 \sqrt{1+x} dx$ D. $\int_2^5 \sqrt{x} dx$
 E. $\int_1^4 \sqrt{1+x} dx$

The term $\frac{3i}{n}$ looks like x_i . when $i = 0$, $x_0 = 0$ and when $i = n$, $x_n = 3$. So we should look for an integral from 0 to 3. The term $\frac{3}{n}$ looks like the length of one of the subintervals, and we are summing $\sqrt{1+x_i}$. So $f(x) = \sqrt{1+x}$. All of this leads us to believe the correct answer is C. If we had set $x_i = 1 + \frac{3i}{n}$, then we would have looked for an integral from 1 to 4. This leads to answer E. However, if $x_i = 1 + \frac{3i}{n}$, then $f(x) = \sqrt{x}$ which eliminates E as a possible answer. So C is the only possible correct choice.

Part 2. Worked out problems.

16. A culture dish containing the bacterium *Rickettsia typhi* and dated April 1 was found in the back of a refrigerator on April 10. There were 8100 cells in the dish. One day later there were 22,000 cells. Assuming the cell population was growing exponentially, how many cells were initially in the culture?

We know that the cell population is given by a function of the form $P(t) = ce^{kt}$. Let $t = 0$ correspond to April 1. Thus, we are looking for the value of c . Note that this means $t = 9$ corresponds to April 10 and $t = 10$ corresponds to April 11. We have the following data

$$\begin{aligned} P(9) &= 8100 = ce^{9k} \\ P(10) &= 22,000 = ce^{10k} \end{aligned}$$

Dividing the second equation by the first equation we have $\frac{22000}{8100} =$

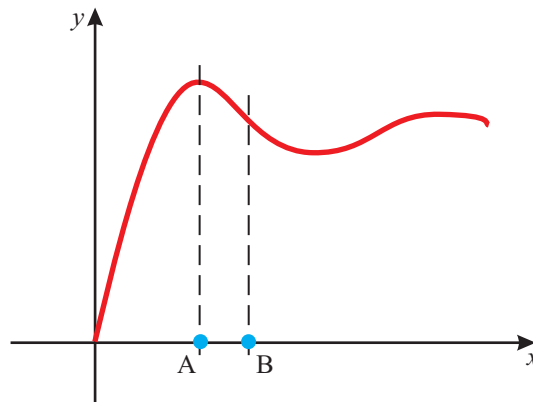
$\frac{e^{10k}}{e^{9k}} = e^k$. Thus,

$$k = \ln \frac{22000}{8100} = \ln \frac{220}{81}$$

Substituting this value of k into the first equation we have

$$\begin{aligned} 8100 &= ce^{9k} \\ c &= 8100e^{-9k} = 8100 \left(e^{\ln(220/81)} \right)^{-9} \\ &= \frac{8100}{\left(\frac{220}{81} \right)^9} \approx 1 \end{aligned}$$

17. $si(x)$ is defined as that function such that $si(0) = 0$ and $si'(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$. A plot of $si(x)$ is shown below.



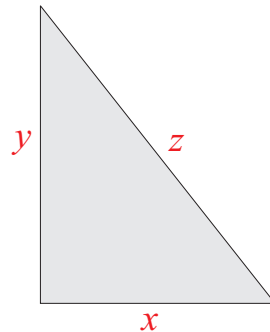
- A. Find the value of x where $si(x)$ is largest. This point is denoted by A on the plot.

$si(x)$ is largest at the first positive value of x for which $\sin x = 0$. Hence, the point A equals π

B. Find the value of the first positive x where $si(x)$ switches from concave down to concave up. This point is denoted by B on the plot.

The second derivative of $si(x) = \frac{d}{dx} \frac{\sin x}{x} = \frac{(\cos x)x - \sin x}{x^2}$. We want the first positive value of x where $x \cos x - \sin x = 0$. This value is approximately 4.493.

18. A right triangle has sides x and y and hypotenuse z . Maximize the sum $2x + y$ under the restrictions $x + z = 1$ and $0 \leq x \leq \frac{1}{4}$.



We have to write the function $2x + y$ in terms of x .

$$\begin{aligned} 2x + y &= 2x + \sqrt{z^2 - x^2} = 2x + \sqrt{(1-x)^2 - x^2} \\ &= 2x + \sqrt{1 - 2x} \end{aligned}$$

The derivative of this function is

$$f'(x) = 2 + \frac{1}{2} (1 - 2x)^{-1/2} (-2) = 2 - \frac{1}{\sqrt{1 - 2x}}$$

Since $0 \leq x \leq 1/4$, we have the following string of inequalities. The only point of which is to see that on the interval $[0, 1/4]$ the derivative is positive. Which means the original function is increasing on the interval $[0, 1/4]$. Hence the maximum value occurs at $x = \frac{1}{4}$, and that maximum value is $2\frac{1}{4} + \sqrt{1 - 2/4} = \frac{1}{2} + \sqrt{1/2} \approx 1.207$.

The lines below verify that $f'(x) > 0$ on $[0, 1/4]$

$$\begin{aligned}0 < x < \frac{1}{4} &\Rightarrow 0 > -2x > -\frac{1}{2} \Rightarrow \\1 > 1 - 2x > \frac{1}{2} &\Rightarrow 1 < \frac{1}{1 - 2x} < 2 \Rightarrow \\1 < \frac{1}{\sqrt{1 - 2x}} < \sqrt{2} &\Rightarrow -1 > \frac{-1}{\sqrt{1 - 2x}} > -\sqrt{2} \Rightarrow \\1 > 2 - \frac{1}{\sqrt{1 - 2x}} > 2 - \sqrt{2} > 0\end{aligned}$$

One could also have just plotted $f(x) = 2x + \sqrt{1 - 2x}$ on the interval $[0, 1/4]$ and ascertained the maximum value occurred at the right endpoint of the interval

19. A. Consider the function $f(x) = 1 + \frac{e^x}{4}$ defined on the interval $[0, 10]$. If we approximate the integral of $f(x)$ over the interval $[0, 10]$ with a right Riemann sum, will this approximation be larger than, equal to, or greater than the actual value of the integral. Be sure to explain your answer.

Since the function $f(x) = 1 + \frac{e^x}{4}$ is increasing and positive on the interval $[0, 10]$ the largest value of $f(x)$ on any subinterval occurs at the right endpoint of the subinterval. Thus, when we use a right Riemann sum the sum of the areas of the rectangles will be larger than the area enclosed by the graph of the function. Sketch the graph of $f(x)$ and plot some rectangles to confirm this.

B. Use a left Riemann sum with 4 equal length subintervals to approximate $\int_1^3 x^2 dx$.

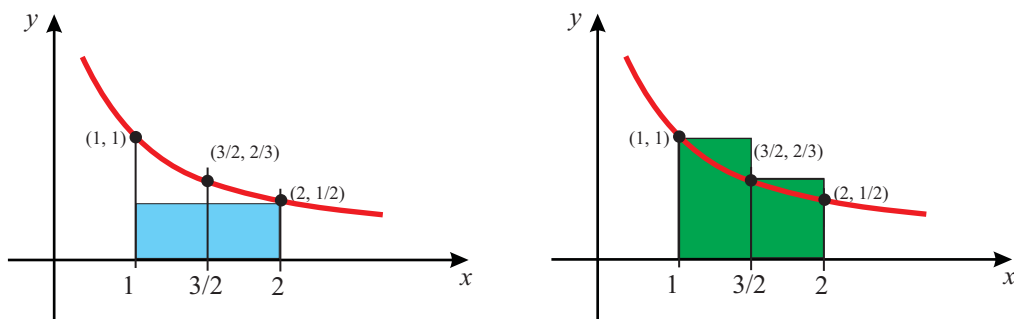
The endpoints of the intervals are 1, 1.5, 2, 2.5, and 3. $f(x) = x^2$. Thus, the left Riemann sum is

$$\begin{aligned}\frac{1}{2}[f(1) + f(1.5) + f(2) + f(2.5)] &= \frac{1}{2}[1 + 2.25 + 4 + 6.25] \\ &= \frac{1}{2}[13.5] = 6.75\end{aligned}$$

20. Without computing the integral show that

$$\frac{1}{2} < \int_1^2 \frac{dx}{x} < \frac{5}{6}.$$

The idea is to use Riemann sums to get upper and lower bounds for the integral $\int_1^2 \frac{dx}{x}$. The picture below shows rectangles for two different Riemann sums.



The blue rectangle represents one Riemann sum. That is there is only one subinterval. Clearly the area of this blue rectangle which is $\frac{1}{2}$ is less than $\int_1^2 \frac{dx}{x}$. The first half of our inequality. The two green rectangles represent a second Riemann sum. This one uses two subintervals, and we see that the combined areas of these two is larger than $\int_1^2 \frac{dx}{x}$. Thus,

$$\int_1^2 \frac{dx}{x} < \frac{1}{2}(1) + \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

This is the second inequality we were to verify.

21. When falling a raindrop experiences the following acceleration

$$a(t) = \begin{cases} -9 + \frac{9}{10}t & \text{if } 0 \leq t \leq 10 \\ 0 & \text{if } 10 < t \end{cases}$$

If the raindrop is 300 meters above the ground with an initial velocity of -6 meters/sec, how long will it take the raindrop to hit the ground. The coordinate system is oriented so that up is positive.

Let $y(t)$ denote the raindrops position at time t . Then we have for $0 \leq t \leq 10$ that

$$\frac{d^2y}{dt^2} = -9 + \frac{9}{10}t$$

Integrating this equation we have $\frac{dy}{dt} = -9t + \frac{9}{20}t^2 + C$. Since the velocity is -6 when $t = 0$, we infer that $C = -6$. Integrating (find an antiderivative) once more we have

$$y(t) = -\frac{9}{2}t^2 + \frac{3}{20}t^3 - 6t + C$$

Since $y(0) = 300$, the constant C must equal 300. Thus, $y(t) = -\frac{9}{2}t^2 + \frac{3}{20}t^3 - 6t + 300$, at least for $0 \leq t \leq 10$. Let's see if the raindrop hits the ground before $t = 10$. That is, does the equation $y(t) = -\frac{9}{2}t^2 + \frac{3}{20}t^3 - 6t + 300 = 0$ have a solution for t between 0 and 10. Using a calculator we see that at $t \approx 8.819$ seconds.