Math 151, Fall 2000
Solutions to Exam 3A

1. What is the domain of the function $f(x) = \sin^{-1}\left(\frac{1}{x}\right)$?

   a) $[-1, 1]$ b) $(-\infty, \infty)$ c) $(-\infty, -1] \cup [1, \infty)$ d) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

   e) $(-\infty, 0) \cup (0, \infty)$

   The domain of the arcsin function is $|x| \leq 1$. Thus, we need $\left|\frac{1}{x}\right| \leq 1$ with $x \neq 0$. This is equivalent to $|x| \geq 1$ or answer c.

2. $\lim_{x \to 0^+} \frac{\ln x}{x} =$

   a) 0 b) 1 c) $-\infty$ d) ∞ e) e

   As x appoaches 0 from above $\ln x$ goes to $-\infty$ and x goes to 0. The ratio is not indeterminant, and must approach $-\infty$ as x approaches 0: answer c.

3. If $f(x) = \ln(\sin x)$, find $f'\left(\frac{\pi}{4}\right)$.

   a) 1 b) $\sqrt{2}$ c) 0 d) $\ln\left(\frac{\ln 2}{2}\right)$ e) does not exist

$$\frac{d}{dx}\ln(\sin x)\bigg|_{x=\pi/4} = \frac{\cos x}{\sin x}\bigg|_{x=\pi/4} = \frac{\cos(\pi/4)}{\sin(\pi/4)} = 1$$

4. $\frac{d}{dx}\left(\tan^{-1}(1 + \tan x)\right) =$

   a) 1 b) 0 c) $\frac{1}{1 + \tan^2 x}$ d) $\frac{1}{\cos^2 x}$ e) $\frac{\sec^2 x}{1 + (1 + \tan x)^2}$

$$\frac{d}{dx}\left(\tan^{-1}(1 + \tan x)\right) = \frac{\sec^2 x}{1 + (1 + \tan x)^2}$$

5. The function f satisfies the equation $f' = 3f$. If $f(0) = 5$, what is the value of $f(5)$?

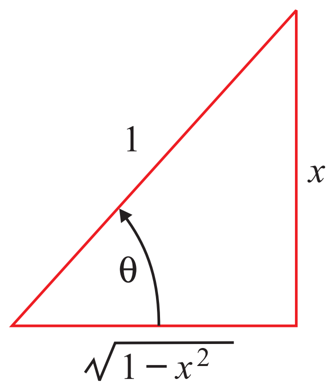
a) 0 b) e^3 c) 15 d) $5e^{15}$ e) none of these

Since $f' = 3f$, we must have $f(x) = ce^{3x}$ for some constant c. And $f(0) = 5$ implies $c = 5$. Thus,

$$f(5) = 5e^{3\times 5} = 5e^{15}$$

6. $\tan\left(\sin^{-1} x\right) =$

a) $\dfrac{1}{\sqrt{1+x^2}}$ b) $\dfrac{x}{\sqrt{1-x^2}}$ c) $\dfrac{x}{\cos x}$ d) $\dfrac{1}{\cos x}$ e) none of these



Let $\theta = \sin^{-1} x$. Then $\sin\theta = x$. Construct a right triangle with side opposite θ of length x and hypotenuse of length 1. Then the side adjascent to θ has length $\sqrt{1-x^2}$, and

$$\tan(\sin^{-1} x) = \tan\theta = \frac{x}{\sqrt{1-x^2}}$$

7. $\displaystyle\lim_{x\to 0} \frac{x}{\ln(1-x)} =$

a) $-\infty$ b) -1 c) 0 d) 1 e) ∞

This ratio has the indeterminate form $\frac{0}{0}$. Hence L'hopital's rule is applicable.

$$\lim_{x\to 0} \frac{x}{\ln(1-x)} = \lim_{x\to 0} \frac{1}{-1/(1-x)} = -1$$

8. Which of the following is the maximum value of $f(x) = x - x^2$, for $-\infty < x < \infty$.

a) 0 b) $\frac{1}{2}$ c) $\frac{1}{4}$ d) $-\frac{1}{4}$ e) none of these

The function $x - x^2$ is differentiable everywhere and its limiting value as x approaches $\pm\infty$ is $-\infty$. Thus, its global maximum must occur when the derivative of the function equals 0.

$$\begin{aligned} f(x) &= x - x^2 \\ f'(x) &= 1 - 2x = 0 \\ x &= 1/2 \\ f\left(\frac{1}{2}\right) &= \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \end{aligned}$$

9. Which of the following is an antiderivative of $\dfrac{x^2 + x + 1}{x}$?

a) $\dfrac{x^3/3 + x^2/2 + x}{x} + C$ b) $1 - \dfrac{1}{x^2} + C$ c) $\dfrac{x^3}{3} + x + C$ d) $\dfrac{x^2}{2} + x + \ln|x| + C$

e) $\dfrac{x^2}{2} + x + 1 + C$

$$\begin{aligned} \int \frac{x^2 + x + 1}{x}\, dx &= \int \left(x + 1 + \frac{1}{x}\right) dx \\ &= \int x\, dx + \int 1 dx + \int \frac{1}{x}\, dx \\ &= \frac{x^2}{2} + x + \ln x + C \end{aligned}$$

10. $\sum_{i=2}^{4} (i-5)^2 =$

a) $\int_2^4 (t-5)^2\, dt$ b) 14 c) $\dfrac{4^3}{3} - \dfrac{2^3}{3}$ d) 26 e) none of these

$$\begin{aligned} \sum_{i=2}^{4} (i-5)^2 &= (2-5)^2 + (3-5)^2 + (4-5)^2 \\ &= 9 + 4 + 1 = 14 \end{aligned}$$

11. Given that $\int_1^3 f(x)\,dx = 4$ and $\int_1^5 f(x)\,dx = 7$ calculate $\int_3^5 (1+2f(x))\;dx$.

a) 8 b) 11 c) 14 d) 22

e) cannot be determined from the given information

$$\begin{aligned}\int_3^5 (1+2f(x))\;dx &= \int_3^5 1\,dx + 2\int_3^5 f(x)\,dx \\ &= 2+2\left(\int_1^5 f(x)\,dx - \int_1^3 f(x)\,dx\right) \\ &= 2+2\,(7-4) = 8\end{aligned}$$

12. Consider the definite integrals

$$I = \int_0^1 e^x \sin^2 x\,dx \quad \text{and} \quad J = \int_0^1 e^x\,dx\,.$$

Which of the following statements is true?

a) $I < 0$ b) $I > J$ c) $I \leq J$ d) $J > e$ e) $J < 1$

Both integrands are non-negative so a) is not true. Since $\sin^2 x < 1$ for x between 0 and 1 we know $e^x \sin^2 x \; < e^x$. Hence b) is false. From the previous reasoning c) is true. In fact, we have $I < J$. Inequalities d) and e) cannot be true either for we have

$$\begin{aligned}J &= \int_0^1 e^x\,dx \leq \int_0^1 e\,dx = e \\ J &= \int_0^1 e^x\,dx \geq \int_0^1 e^0\,dx = 1\end{aligned}$$

Part 2. Worked out problems.

13. (8) In a chemical reaction a compound C decomposes at a rate proportional to the amount of C that remains. It is found by experiment that 8 grams of C diminish to 4 grams in 2 hours. How long will it take 8 grams of C to diminish to 1.5 grams?

From the fact that $\frac{dC}{dt} = kC$, we know that $C(t) = ce^{kt}$. We are told that the initial amount of C is 8. Thus, $8 = C(0) = ce^0 = c$. We also know that $C(2) = 4$. Thus,

$$\begin{aligned} 4 &= C(2) = 8e^{2k} \\ \ln\frac{1}{2} &= 2k \\ k &= \frac{\ln 1/2}{2} \end{aligned}$$

Hence, $C(t) = 8e^{t\ln(1/2)/2}$. The next step is to determine for which value of t, $C(t) = 1.5$.

$$\begin{aligned} 8e^{t\ln(1/2)/2} &= 1.5 \\ e^{t\ln(1/2)/2} &= \frac{1.5}{8} \\ \frac{t}{2}\ln(1/2) &= \ln(1.5/8) \\ t &= 2\frac{\ln(1.5/8)}{\ln(1/2)} \approx 4.83 \end{aligned}$$

14. (8) Compute $\lim\limits_{x\to\infty} x^{-3/\sqrt{x}}$. Be sure to justify your work. Computing the limit with the aid of your calculator is not sufficient.

$$\lim_{x\to\infty} x^{-3/\sqrt{x}} = \lim_{x\to\infty} e^{-3\ln x/\sqrt{x}} = e^{-\lim\limits_{x\to\infty}(\ln x/\sqrt{x})}$$

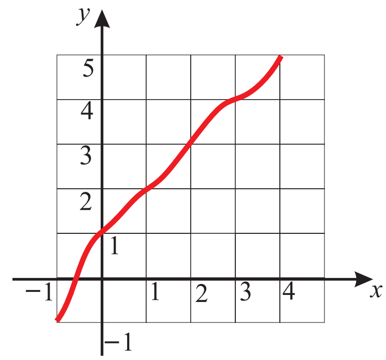
Thus, we need to compute

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty}\frac{1/x}{(1/2)x^{-1/2}} = \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0$$

So we have

$$\lim_{x\to\infty} x^{-3/\sqrt{x}} = \lim_{x\to\infty} e^{-3\ln x/\sqrt{x}} = e^{-\lim\limits_{x\to\infty}(\ln x/\sqrt{x})} = e^{-0} = 1$$

15. (9) The plot of a function $f(x)$ is shown at the right. Use it to answer the following questions.



(a) Using a right Riemann sum (right endpoint evaluation) with 2 equal length subintervals approximate $\int_0^4 f(x)\,dx$.

$$\begin{aligned}\int_0^4 f(x) &\approx 2\,(f(2)+f(4)) \\ &= 2\,(3+5) = 16\end{aligned}$$

(b) Using a left Riemann sum (left endpoint evaluation) with 2 equal length subintervals approximate $\int_0^4 f(x)\,dx$.

$$\begin{aligned}\int_0^4 f(x) &\approx 2\,(f(0)+f(2)) \\ &= 2\,(1+3) = 8\end{aligned}$$

(c) Let RR denote the approximate value you found in part a). Explain why one of the following inequalities is false and the other is true.

$$RR < \int_0^4 f(x)\,dx \quad RR > \int_0^4 f(x)\,dx$$

The function $f(x)$ is increasing on the interval $[0,4]$. Thus, the value of f at a right endpoint is greater than or equal to the value of f at any point in the corresponding subinterval. Moreover since the function is strictly increasing we have the area contained by one of the constructed rectangles strictly greater than the area underneath the graph of f. Thus,

$$RR > \int_0^4 f(x)\,dx\,.$$

16. (9) Let $f(x) = \frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{1}{2}x^2 + 2x$. Be sure to explain your work. A graph of f will not be sufficient justification. The values $x = -1,\ 1,\ 2$ are critical numbers of $f(x)$.

(a) Find all intervals where f is increasing.

$$\begin{aligned} f'(x) &= x^3 - 2x^2 - x + 2 \\ &= (x+1)(x-1)(x-2) \end{aligned}$$

Since the derivative of f is positive on the intervals $(-1, 1)$ and $(2, \infty)$, f is increasing on these two intervals.

(b) Find all intervals where f is concave down.

$$f''(x) = 3x^2 - 4x - 1$$

The second derivative of f is a quadratic polynomial. It is zero when $x = \frac{2 \pm \sqrt{7}}{3}$, and it is negative between the two roots. Thus, f is concave down on the interval $\left(\frac{2-\sqrt{7}}{3}, \frac{2+\sqrt{7}}{3}\right)$.

(c) Find all x's where f has a local minimum.

From the answer to part a. we know that f is decreasing on $(-\infty, -1)$, increasing on $(-1, 1)$, decreasing on $(1, 2)$ and then increasing on $(2, \infty)$. This tells us that f has a local minimum at $x = -1$ and 2.

(d) Find all points of inflection of f.

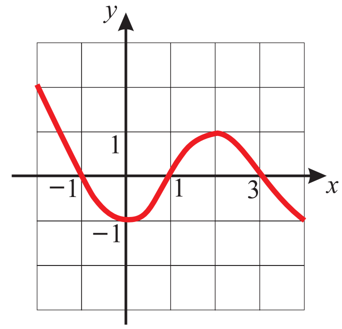
f has points of inflection where the concavity of f changes or, in this case, when the sign of the second derivative changes. That is, at $x = \frac{2 \pm \sqrt{7}}{3}$. So the points of inflection occur at the two points

$$\left(\frac{2-\sqrt{7}}{3}, f\left(\frac{2-\sqrt{7}}{3}\right)\right) \text{ and } \left(\frac{2+\sqrt{7}}{3}, f\left(\frac{2+\sqrt{7}}{3}\right)\right)$$

or in decimal form

$(-0.215,\ -0.446)$ and $(1.548,\ 0.860)$

17. (9) The plot to the right is the graph of the derivative, f', of f. Use the plot to answer the following questions.



A plot of the **derivative** of f.

(a) On what intervals is f increasing?

According to the graph, the derivative of f is positive on the intervals $(-2, -1)$ and $(1, 3)$. Thus, f is increasing on those intervals.

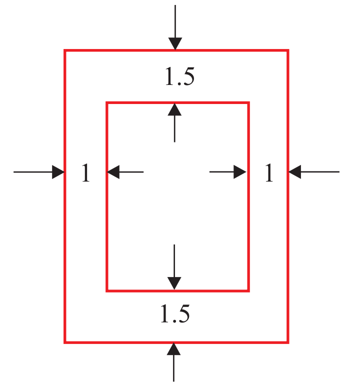
(b) On what intervals is f concave down?

f is concave down when its derivative is decreasing. From the graph the derivative of f is decreasing on the intervals $(-2, 0)$ and $(2, 4)$. So f is concave down on these intervals too.

(c) For what values of x does f have a local maximum?

f will have a local minimum when it changes from increasing to decreasing as x increases. This occurs at the points $x = -1$ and $x = 3$.

18. (9) A rectangular poster is to have an area of 180 in^2 with 1.5 inch margins at the top and bottom, and a 1 inch margin on the sides. What poster dimensions will give the largest printed area? The printed area consists of the area of the poster minus the area of the margins. Be sure to justify your answer.



Let x denote the width of the poster, y the height of the poster, and A the area of the printed region. Then

$$\begin{aligned} A &= (x-2)(y-3) \\ &= (x-2)\left(\frac{180}{x}-3\right) \\ &= 180 - \frac{360}{x} - 3x + 6 \end{aligned}$$

The domain of this function is $x \geq 2$ and $y \geq 3$. In terms of x we have $\frac{180}{x} \geq 3$ or $x \leq 60$. Thus, the function we have to maximize is

$$A(x) = 180 - \frac{360}{x} - 3x + 6, \quad \text{for } 2 \leq x \leq 60$$

A quick check shows that $A(2) = A(60) = 0$, and that the function is positive on the open interval $(2, 60)$. Thus, its maximum must occur at a critical point. Since $A(x)$ is differentiable everywhere except $x = 0$, we are looking for points where the derivative of A is zero.

$$\begin{aligned} A'(x) &= \frac{360}{x^2} - 3 = 0 \\ x^2 &= \frac{360}{3} = 120 \\ x &= \sqrt{120} = 2\sqrt{30} \end{aligned}$$

Thus, the poster should have dimensions

$$\begin{aligned} \text{width} &= x = 2\sqrt{30} \approx 10.95 \\ \text{height} &= y = \frac{180}{x} = \frac{180}{2\sqrt{30}} = 3\sqrt{30} \approx 16.432 \end{aligned}$$