Spring 1998

Maple Lab, Week 27

Projectile Drag Studied by Perturbation Theory

Maple tips for this lab:

1. The commands `taylor` and `convert`, described in Sec. 10.5 of *CalcLabs*, may be helpful.

2. Work in terms of Maple expressions, rather than Maple functions, as much as possible. (If you do define a function, you need to consider whether it should be a function of t, of k, or of both.)

3. Most of the differential equations that arise here are merely antiderivative problems, so they are more easily solved with `int` than with `dsolve`.

4. If Maple evaluates an integral in terms of an unfamiliar function such as "arctanh", consult Sec. 6.7 of Stewart, especially the two boxes on page 416! (arctanh is the same thing as $\tanh^{-1}$.) Also, the command `convert(",ln);` may help.

Review of the problem and the solution without drag: We return to the projectile problem of Lab 25. At $t = 0$ the projectile is fired from $(x, y) = (0, 0)$ with initial velocity $\vec{\mathbf{v}}(0) = \langle v_x(0), v_y(0)\rangle = v_\mathrm{i}\langle\cos\theta, \sin\theta\rangle$, where $v_\mathrm{i} \equiv |\vec{\mathbf{v}}(0)|$ is the initial speed. Assume that $v_x(0)$ and $v_y(0)$ are both strictly positive (in other words, $v_\mathrm{i} > 0$ and $0 < \theta < \frac{\pi}{2}$). *Note:* θ is a *constant*, parametrizing the initial data; it is *not* the polar coordinate of the vector $\vec{\mathbf{v}}(t)$, which is a function of t. The subscript "i" stands for "initial", "f" stands for "final", and "0" stands for "zeroth-order" — that is, the solution without drag.

When drag is neglected, the solution for the components of position and velocity is

$$x_0(t) = v_\mathrm{i}\cos\theta\, t, \qquad y_0(t) = v_\mathrm{i}\sin\theta\, t - \tfrac{1}{2}gt^2,$$

$$v_{x0}(t) = v_\mathrm{i}\cos\theta, \qquad v_{y0}(t) = v_\mathrm{i}\sin\theta - gt.$$

Where (and when) does the projectile land? Setting $y_0(t_\mathrm{f}) = 0$, we find that

$$t_\mathrm{f} = \frac{2v_\mathrm{i}\sin\theta}{g}, \qquad x_\mathrm{f} \equiv x_0(t_\mathrm{f}) = \frac{2v_\mathrm{i}^2\sin\theta\cos\theta}{g} = \frac{v_\mathrm{i}^2\sin(2\theta)}{g}.$$

Therefore, given x_f, you can find v_i and θ that will enable you to hit that target. (In fact, you can hold θ fixed and vary v_i, or you can hold v_i fixed (if it's sufficiently large) and vary θ, depending on the design of your launcher.)

Exercises: From Lab 25, the equations of motion with atmospheric drag included are

$$x' = v_x\,, \quad y' = v_y\,, \quad v_x' = -\tfrac{k}{m} v_x |\vec{\mathbf{v}}|, \quad v_y' = -\tfrac{k}{m} v_y |\vec{\mathbf{v}}| - g.$$

A "perturbative" solution is an approximate solution constructed as a Taylor series in k; in practice, this usually means just the first 2 or 3 terms in the Taylor series. To construct a first-order (2-term) solution we write

$$x(t) = x_0(t) + \tfrac{k}{m} x_1(t), \qquad v_x(t) = v_{x0}(t) + \tfrac{k}{m} v_{x1}(t),$$

and similar formulas for y and v_y. The zeroth-order terms will turn out to be the solution without drag; the first-order terms are unknowns to be found.

1. Using an appropriate Maple syntax, substitute the assumed form of the solution into the equations of motion and expand out the right-hand sides as first-order Taylor polynomials in k. (It is best to think of this as a hand calculation with Maple assistance. In particular, it is not necessary to take t derivatives with Maple, nor to indicate the "(t)" argument explicitly.) Verify that the zeroth-order terms cancel out if they are evaluated from the drag-free solution given above. Divide the rest of the equations by k to get differential equations satisfied by the first-order terms. For example, one of your four equations should turn out to be

$$v_{x1}' = -v_\mathrm{i} \cos\theta \sqrt{v_\mathrm{i}^2 - 2 g v_\mathrm{i} \sin\theta\, t + g^2 t^2}.$$

2. Solve for $v_{x1}(t)$ and $v_{y1}(t)$, and hence for $x_1(t)$ and $y_1(t)$. The initial values $x_1(0)$ and $y_1(0)$ should be 0 (since we are still firing from the origin), but the initial values $v_{x1}(0)$ and $v_{y1}(0)$ are arbitrary constants of integration.

3. As in Lab 25, consider the case

$$v_\mathrm{i} = 100, \quad \theta = \frac{\pi}{4}, \quad \frac{k}{m} = 0.0025.$$

Plot together the three trajectories: (1) the zeroth-order solution, (x_0, y_0); (2) the first-order solution, $(x_0 + \frac{k}{m} x_1, y_0 + \frac{k}{m} y_1)$; (3) the numerical solution you found in Lab 25.

Accepting the numerical solution as approximately valid, pass judgment on the accuracy of the perturbative solution.

4. Accepting the perturbative solution as valid (regardless of the outcome of Step 3), find how to modify v_i and θ (i.e., choose $v_{x1}(0)$ and $v_{y1}(0)$) so that the projectile will hit the desired spot x_f. As previously remarked, there is more than one way of doing this, so you will probably want to choose the one that corresponds to your project design.

Notice that the result has one huge advantage over the numerical solution: It is a formula that can be used over and over for various values of x_{f} and $\frac{k}{m}$, whereas the numerical solution has to be recomputed each time. Unfortunately, the perturbative solution is not accurate unless both $\frac{k}{m}$ and t are sufficiently small. (And unfortunately, safety considerations compel your project to use Ping-Pong balls, rather than, say, bullets.)

Extra credit exercises:

5. Find a **second-order** solution,

$$x = x_0 + \tfrac{k}{m} x_1 + \left(\tfrac{k}{m}\right)^2 x_2 \,, \quad \text{etc.}$$

Is the increased accuracy worth the effort?

6. How does a projectile behave in the thick atmosphere of the tiny planet Zarf? (That is, g is small but k is not.) To get the zeroth-order solution, ignore gravity but not drag. It is physically obvious that the zeroth-order motion is in a straight line [explain why], so in this case $v_{y0}(t)/v_{x0}(t)$ is a constant, equal to $\tan\theta$. This observation will enable you to get a separable differential equation for $v_{x0}(t)$. Then construct a first-order solution, $x = x_0 + g x_1$, etc.

7. In the one-dimensional version of this problem,

`http://www.math.tamu.edu/~fulling/coalweb/airdrag.dvi`,

to avoid a complication it was necessary to assume that the initial velocity was negative. Why does that complication not arise in the two-dimensional problem for reasonable initial data? (Firing the launcher straight up is considered unreasonable.)

8. Solve the one-dimensional problem exactly, and compare with the perturbative solution. (The differential equation for the velocity is separable, but may require consulting an integral table or Sec. 6.7 or Sec. 7.4.) You should find that as $t \to \infty$ the velocity approaches the limit $-\sqrt{gm/k}$.