Final Examination – Solutions

Name: ____________________________ Number: ________________
(as on attendance sheets)

Calculators may be used for simple arithmetic operations only!

1. (15 pts.) Let \(a_n\) be the number of ways to distribute \(n\) apples to 3 children so that no child gets fewer than 2 apples and no child gets more than 5 apples. Find the generating function for \(a_n\) and use it to compute \(a_{10}\).

The generating function for the possible results for one child is \(x^2 + x^3 + x^4 + x^5\).

The generating function for the problem is the cube of that,

\[
(x^2 + x^3 + x^4 + x^5)^2 = x^6(1 + x + x^2 + x^3)^3
= x^6 \left(\frac{1 - x^4}{1 - x}\right)^3.
\]

We need the coefficient of \(x^{10}\) in its Maclaurin expansion, hence the coefficient of \(x^4\) in

\[
\left(\frac{1 - x^4}{1 - x}\right)^3 = (1 - x^4)^3 \sum_{j=0}^{\infty} \binom{3}{j} x^j
= [1 - 3x^4 + O(x^8)] \left[ 1 + 3x + \cdots + \frac{3 \cdot 4 \cdot 5 \cdot 6}{4!} x^4 + \cdots \right]
= \cdots + (-3 + 15)x^4 + \cdots.
\]

Thus \(a_{10} = 12\). (Here \(O(x^8)\) stands for the irrelevant terms of degree 8 or higher; \(O\) refers here to the asymptotic behavior near 0, not \(\infty\).)

2. (40 pts.) Solve these recursion relations.

   (a) \(a_n = a_{n-1} + 2a_{n-2}\), \(a_0 = 5\), \(a_1 = 1\).

Try \(a_n = r^n\), getting the condition

\[
0 = r^2 - r - 2 = (r + 1)(r - 2).
\]

Thus the general solution is \(a_n = b_1(-1)^n + b_22^n\). From the initial data, we must have

\[
5 = b_1 + b_2, \quad b_1 = 3,
1 = -b_1 + 2b_2; \quad b_2 = 2.
\]

Thus

\[
a_n = 3(-1)^n + 2 \cdot 2^n = 3(-1)^n + 2^{n+1}.
\]
(b) \( a_n = 6a_{n-1} \), \( a_0 = 7 \).

\[ a_n = 7 \cdot 6^n. \]

(c) \( a_n = 6a_{n-1} - 9a_{n-2} \). [Find the general solution.]
In the usual way, we get
\[ 0 = r^2 - 6r + 9 = (r - 3)^2. \]
Since there is only one root, we must multiply the solution \( r^n \) by \( n \) to get a second independent solution. The general solution is
\[ a_n = b_13^n + b_2n3^n. \]

(d) \( a_n - a_{n-1} = n \), \( a_0 = 0 \).
Method 1: Temporarily write \( n \) as \( j \):
\[ a_j - a_{j-1} = j. \] Sum this equation from \( j = 1 \) to \( j = n \):
\[ a_n - a_0 = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}. \]
Thus \( a_n = \frac{1}{2}n(n+1) \).

Method 2: First solve the corresponding homogeneous equation, \( a_n^h - a_{n-1}^h = 0 \). It’s obvious that the solution is independent of \( n \), and indeed the official maneuver of trying \( a_n^h = r^n \) yields \( r - 1 = 0 \), hence \( a_n^h = r^1 = 1 \) (times an arbitrary constant). If we now seek a particular solution of the nonhomogeneous equation, we must start with \( An + B \) (because the nonhomogeneous term contains \( n \)) and then promote it to
\[ a_n = An^2 + Bn \]
(because the homogeneous solution is \( 1 \)). Substituting into the recursion relation, we find
\[ n = An^2 + Bn - A(n-1)^2 - B(n-1) \]
\[ = A(2n-1) + B \]
\[ = 2An + (B - A). \]
Thus
\[ 2A = 1, \quad A = \frac{1}{2}, \]
\[ B - A = 0; \quad B = \frac{1}{2}. \]
Thus \( a_n = \frac{1}{2}n^2 + \frac{1}{2}n + C \). The initial condition \( 0 = a_0 = C \) gives us (the hard way) the same solution as before.

3. (24 pts.) Let \( O(n) \) stand for “\( n \) is an odd integer”. Consider the statement

\( p: \) If the product of two integers is odd, then both integers are odd.

(a) Rewrite \( p \) in symbols (\( O \), quantifiers, and logical connectives).
\[ \forall m \forall n [O(mn) \to O(m) \land O(n)]. \]
(b) State the converse of p both in symbols and in smooth English.

\[ \forall m \forall n [\mathcal{O}(m) \land \mathcal{O}(n) \rightarrow \mathcal{O}(mn)] . \]

The product of two odd integers is always odd.

(Of course, considerable variation in the English sentences is possible.)

(c) State the negation of p both in symbols and in smooth English.

\[ \exists m \exists n [\mathcal{O}(mn) \land \neg(\mathcal{O}(m) \land \mathcal{O}(n))] . \]

(Rewriting the last clause as \( \neg\mathcal{O}(m) \land \neg\mathcal{O}(n) \) is optional.)

There exist two integers, not both odd, whose product is odd.

(d) Which of those three statements is/are true?

The proposition and its converse are true. (“Odd” means “not divisible by 2.”) The negation is false.

4. (14 pts.) Show (by any method) that \( \sum_{k=0}^{700} \binom{700}{k} = 2^{700} \).

You’ve seen this one before. See the solutions for Test A, Question 1(b).

5. (28 pts.) For each of these “divide and conquer” recursions, either find an asymptotic estimate on \( T(n) \) using the master theorem, or explain why the master theorem does not apply. \( \text{Hint: What is } 4^{3/2} ? \)

(a) \( T(n) = 3T\left(\frac{n}{2}\right) + n \).

Because \( L \equiv \log_2 3 > 1 \), one has \( s(n) \equiv n \in O(n^{L-\epsilon}) \), so \( T \in \Theta(n^L) \) by Part 1 of the theorem.

(b) \( T(n) = 8T\left(\frac{n}{4}\right) + \frac{n\sqrt{n}}{\lg n} \).

\( L \equiv \log_4 8 = \frac{3}{2} \), since \( 4^{3/2} = 2^3 = 8 \). Now we observe that \( s(n) \equiv n^{L/\lg n} \notin \Theta(n^L) \), but \( s(n) \notin O(n^{L-\epsilon}) \) either. Therefore, the theorem doesn’t apply.

(c) \( T(n) = 8T\left(\frac{n}{4}\right) + \frac{n\sqrt{n}}{2} \).

Again \( L = \frac{3}{2} \). This time \( s(n) \equiv \frac{1}{2}n^L \in \Theta(n^L) \), so \( T \in \Theta(n^{3/2} \lg n) \) by Part 2 of the theorem.

6. (20 pts.) [Leave answers in terms of factorials.]

(a) How many permutations of the English alphabet (26 letters) contain the block YES? Treat YES as a single letter. There are 23 other letters. The number of permutations is \( 24! \).
(b) What is the probability that a randomly chosen permutation of the alphabet contains neither YES nor NO?

24! permutations contain YES, and by the analogous argument 25! contain NO. But some of these contain both, and we don’t want to count those twice. If YES and NO are treated as single letters, there are $26 - 5 + 2$ letters and hence 23! such permutations. Therefore, the probability of avoiding YES and NO is

$$\frac{26! - 25! - 24! + 23!}{26!}.$$ 

7. (15 pts.) Define a sequence by

$$t_0 = 1, \quad t_1 = 2, \quad t_2 = 4, \quad t_n = t_{n-1} + t_{n-2} + t_{n-3} \text{ for } n \geq 3.$$ 

Prove by induction that $t_n \geq 1.5^n$ for all $n \geq 0$.

*Base:* $1.5^0 = 1 = t_0, \quad 1.5^1 = \frac{3}{2} < 2 = t_1, \quad 1.5^2 = \frac{9}{4} < 4 = t_2$.

*Induction:* For $n > 2$,

$$t_n = t_{n-1} + t_{n-2} + t_{n-3} \geq 1.5^{n-1} + 1.5^{n-2} + 1.5^{n-3}$$

$$= 1.5^{n-3} \left(1 + \frac{3}{2} + \frac{9}{4}\right)$$

$$= 1.5^{n-3} \cdot \frac{19}{4}$$

$$\geq 1.5^{n-3} \cdot \frac{27}{8} = 1.5^n.$$

8. (20 pts.) Let $F$ be this set of functions:

$$\left\{ n + \ln n, \ 2^n, \ \sqrt{n^3 + 1}, \ \frac{n^2 + 1}{n + 2}, \ 4^n, \ \frac{n^2 + 1}{\sqrt{n}} \right\}$$

Let $R$ be the equivalence relation defined on $F$ by

$$(f, g) \in R \iff f \in \Theta(g).$$

(a) What is the partition of $F$ induced by $R$?  [Answer should be a list of subsets ("cells") of $F$.

Note that $2^n = e^{n \ln 2}$ and $4^n = e^{n \ln 4}$ are in separate cells. The other two cells are easily identified as $\Theta(n)$ and $\Theta(n^{3/2})$. So the partition is

$$\left\{ n + \ln n, \frac{n^2 + 1}{n + 2} \right\}, \quad \{2^n\}, \quad \{4^n\}, \quad \left\{ \sqrt{n^3 + 1}, \ \frac{n^2 + 1}{\sqrt{n}} \right\}.$$
(b) Take one representative from each cell of this partition and order these representatives from the slowest to the fastest (in terms of growth at infinity).
\[ n + \ln n < \sqrt{n^3 + 1} < 2^n < 4^n \]

9. (24 pts.) Let \( A \) be a set and let \( \mathcal{U} \) be the set of all subsets of \( A \). (\( \mathcal{U} \) is sometimes denoted by \( \mathcal{P}(A) \) or \( 2^A \).) Define a relation \( R \) on \( \mathcal{U} \) by
\[
(X, Y) \in R \iff X \cap Y = \emptyset.
\]

Answer the follow questions, with justifications:

(a) Is \( R \) reflexive?
NO — \( X \cap X = \emptyset \) is never true except for \( X = \emptyset \).

(b) Is \( R \) antisymmetric?
NO — \( X \cap Y = \emptyset \) requires, rather than forbids, \( Y \cap X = \emptyset \).

(c) Is \( R \) symmetric?
YES — see justification for (b).

(d) Is \( R \) transitive?
NO — if \( X \neq \emptyset \) but \( X \cap Y = \emptyset \), then \( Y \cap X = \emptyset \), but \( X \cap X = \emptyset \) does not follow.

(e) Is \( R \) an equivalence relation?
NO — it is not reflexive nor transitive.

(f) Is \( R \) an ordering?
NO — it is not antisymmetric nor transitive, so it isn’t a strict ordering (not to mention an inclusive one, which also requires reflexivity).

Remark: Above we have tacitly assumed that \( A \neq \emptyset \). In the trivial special case where \( A \) itself is the empty set, the answer to all the questions is YES!