## Final Examination - Solutions

Name: $\qquad$ Number: $\qquad$
(as on attendance sheets)

## Calculators may be used for simple arithmetic operations only!

1. (30 pts.) Solve these recursion relations.
(a) $a_{n+2}-3 a_{n+1}+2 a_{n}=0, \quad a_{0}=2, \quad a_{1}=1$.

This is a linear recursion with constant coefficients, so try the solution $a_{n}=r^{n}$. Cancelling $r^{n}$, we arrive at the equation $0=r^{2}-3 r+2=(r-1)(r-2)$. So $r=1$ and $r=2$ should work. The general solution is $a_{n}=c_{1} 1^{n}+c_{2} 2^{n}$. We must have

$$
2=a_{0}=c_{1}+c_{2}, \quad 1=a_{1}=c_{1}+2 c_{2} .
$$

Solve: $c_{1}=3, c_{2}=-1$. Thus the solution to the recursion is

$$
a_{n}=3-2^{n} .
$$

(b) $a_{n+1}=4 a_{n}+3, \quad a_{0}=-1$.

Method 1: Rewrite this as

$$
\begin{aligned}
a_{n}-4 a_{n-1} & =3, \\
4\left(a_{n-1}-4 a_{n-2}\right) & =4 \cdot 3, \\
& \cdots \\
4^{n}\left(a_{1}-4 a_{0}\right) & =4^{n} \cdot 3 .
\end{aligned}
$$

Add: $\quad a_{n}-4^{n+1} a_{0}=3 \sum_{k=1}^{n} 4^{k}=3 \cdot \frac{4^{n-1}-1}{4-1}$. Thus $a_{n}=-4^{n-1}+4^{n-1}-1=-1$. The solution is (perhaps surprisingly)

$$
a_{n}=-1 \text { for all } n .
$$

Method 2: Let's see, ...

$$
a_{1}=4 \cdot(-1)+3=-1, \quad a_{2}=4 \cdot(-1)+3=-1, \ldots .
$$

By induction, if $a_{n}=-1$, then $a_{n+1}=4 \cdot(-1)+3=-1$, so $a_{n}=-1$ for all $n$.
2. (10 pts.) Simplify $\left[2 x^{3}+3 x^{2}+O(x)\right]\left[e^{2 x}+3 e^{x}+O\left(\frac{e^{x}}{\ln x}\right)\right]$. (Answer should have the structure " $f(x)+O(g(x))$ " where $f(x)$ and $g(x)$ have just one term each.)
The leading term should be the product of the fastest-growing terms in the two factors: $f(x)=$ $2 x^{3} e^{2 x}$. A serious candidate for the next term must be the product of the leading term of one factor times the second-biggest term of the other factor: $g(x)=6 x^{3} e^{x}$ or $3 x^{2} e^{2} x$. The second of these is the dominant one, and the numerical factor is irrelevant to " $O$ ", so we can write

$$
2 x^{3} e^{2 x}+O\left(x^{2} e^{2 x}\right)
$$

3. (25 pts.)
(a) Prove that $\binom{m+1}{k}=\binom{m}{k}+\binom{m}{k-1}$ (for $\left.k=1, \ldots, m\right)$.

Method 1: $\binom{m}{k}+\binom{m}{k-1}=\frac{m!}{k!(m-k)!}+\frac{m!}{(k-1)!(m-k+1)!}=m!\frac{(m-k+1)+k}{k!(m-k+1)!}=\frac{(m+1)!}{k!(m-k+1)!}$, QED.
Method 2: $\binom{m+1}{k}$ is the number of $k$-element subsets of a set with $m+1$ elements, say

$$
\left\{a_{1}, \ldots, a_{m}, a_{m+1}\right\}
$$

Such a subset may either be one of the $\binom{m}{k} k$-element subsets of $\left\{a_{1}, \ldots, a_{m}\right\}$, or be formed by adding $a_{m+1}$ to one of the $\binom{m}{k-1}(k-1)$-element subsets of $\left\{a_{1}, \ldots, a_{m}\right\}$.
(b) Using (a), give a proof by induction of the binomial theorem,

$$
(a+b)^{m}=\sum_{k=0}^{m}\binom{m}{k} a^{k} b^{m-k} \quad(\text { for all } m \in \mathbf{N})
$$

Base: If $n=0$, the assertion is $(a+b)^{0}=\binom{0}{0} a^{0} b^{0}$, or $1=1$. (If $n=1$, the assertion is $a+b=a+b$. So far, so good.)
Induction: Multiply the equation by $(a+b)$ :

$$
\begin{aligned}
(a+b)^{m+1} & =\sum_{k=0}^{m}\binom{m}{k} a^{k} b^{m-k}(a+b) \\
& =\sum_{k=0}^{m}\binom{m}{k} a^{k+1} b^{m-k}+\sum_{k=0}^{m}\binom{m}{k} a^{k} b^{m-k+1} \\
& =\sum_{j=1}^{m+1}\binom{m}{j-1} a^{j} b^{m-j+1}+\sum_{k=0}^{m}\binom{m}{k} a^{k} b^{m+1-k} \\
& =\sum_{k=1}^{m}\left[\binom{m}{k-1}+\binom{m}{k}\right] a^{k} b^{m+1-k}+\binom{m}{m} a^{m+1} b^{0}+\binom{m}{0} a^{0} b^{m+1} \\
& =\sum_{k=1}^{m}\binom{m+1}{k} a^{k} b^{m+1-k}+\binom{m+1}{m+1} a^{m+1} b^{0}+\binom{m+1}{0} a^{0} b^{m+1} \\
& =\sum_{k=0}^{m+1}\binom{m+1}{k} a^{k} b^{m+1-k},
\end{aligned}
$$

which is what we need to prove. (The next-to-last step uses the fact that $\binom{m}{0}=1=\binom{m}{m}$ for all $m$. This special treatment of the first and last terms is not necessary if one defines $\binom{m}{k}$ to be 0 for $k=-1$ and $k=m+1$, so that (a) remains true when $k=0$ or $m+1$.)
4. (20 pts.) There are 33 students in this class. What is the probability that at least 2 of you have the same birthday? (Ignore Feb. 29. Working out the arithmetic is not required.) Hint: If $|A|=33$ and $|B|=365$, how many one-to-one functions $f: A \rightarrow B$ are there?
There are $P(365,33)=365 \times 364 \times(365-33+1)=\frac{365!}{332!}$ injective functions. There are $365^{33}$ functions in all. Two people have the same birthday precisely if the birthday function from the class to the days of the year is not injective. The probability that this happens is

$$
p=1-\frac{P(365,33)}{365^{33}} \quad \text { or } \quad \frac{365^{33}-P(365,33)}{365^{33}} \quad \text { or } \quad 1-\frac{364}{365} \times \frac{363}{365} \times \cdots \times \frac{333}{365} .
$$

This number is approximately 0.7 .
5. (20 pts.) "In a certain Pascal program, if the integer variable $n$ is initialized in the program, then $n>0$ throughout program execution. There is a WRITE statement that prints the value of $n^{2}+n$ if $n>0$. If the value of $n$ is supplied by the user in a READ statement, then $n$ is not initialized in the program. The value of $n^{2}+n$ was printed when Ron ran the program. Therefore, Ron did not supply a value for the integer variable $n$." Write the foregoing argument in symbolic form. (Define statement variables
$p: \quad n$ is initialized in the program,
etc.) Then provide a proof for the argument or give a counterexample (of truth-value assignments to your statement variables) to show that it is invalid.
Define the statement variables
$p$ : $\quad n$ is initialized in the program
$q: \quad n>0$ (at the time of the WRITE)
$r: \quad n^{2}+n$ is printed
$s$ : The user supplied a value for $n$
The structure of the argument is

$$
\begin{aligned}
& p \rightarrow q \\
& q \rightarrow r \\
& s \rightarrow \neg p \\
& r \\
& \hline \therefore \neg s
\end{aligned}
$$

This argument is NOT valid; it seems to be "arguing the converse" (confusing $p \rightarrow r$ with $r \rightarrow p$ or $\neg p \rightarrow \neg r$ ). An assignment of truth values that makes all the hypotheses true and the conclusion false is

$$
s, \quad \neg p, \quad r, \quad \text { and either } q \text { or } \neg q .
$$

6. (20 pts.) Apply the Master Theorem (copy provided) to determine the asymptotic growth of the solutions of these recursions.
(a) $T(n)=2 T\left(\frac{n}{2}\right)+3 \lg n$

Here $a=2, b=2$, so $\log _{b} a=1$. Since $3 \lg n \in O\left(n^{1-\epsilon}\right)$, Case 1 applies, and $T \in \Theta(n)$.
(b) $F(n)=4 F\left(\frac{n}{2}\right)+n^{3}$

Here $a=4, b=2$, so $\log _{b} a=2$. Since $n^{3} \in \Omega\left(n^{2+\epsilon}\right)$ and $4(n / 2)^{3}=n^{3} / 2$, Case 3 applies, and $F \in \Theta\left(n^{3}\right)$.
7. (20 pts.) For a function $f: A \rightarrow B$, and subsets $B_{1}, B_{2} \in B$,
(a) Explain what the "preimage" notation, $f^{-1}\left(B_{j}\right)$, means.

For $a \in A, a \in f^{-1}\left(B_{j}\right)$ means that there is a $b \in B_{j}$ for which $f(a)=b$. In other words, $f^{-1}\left(B_{j}\right)$ is the set of all elements of $A$ that get mapped into $B_{j}$ by $f$.
(b) Prove that $f^{-1}\left(B_{1} \cap B_{2}\right)=f^{-1}\left(B_{1}\right) \cap f^{-1}\left(B_{2}\right)$.

Suppose that $a \in f^{-1}\left(B_{1} \cap B_{2}\right)$. Then there is a $b$ in $B_{1} \cap B_{2}$ with $f(a)=b$. Since $b$ is in $B_{1}$ and in $B_{2}, a$ is in $f^{-1}\left(B_{1}\right)$ and in $f^{-1}\left(B_{2}\right)$, hence in their intersection. So far we have proved that the left side is a subset of the right side. Now suppose that $a \in f^{-1}\left(B_{1}\right) \cap f^{-1}\left(B_{2}\right)$. Then there exist $b_{1} \in B_{1}$ and $b_{2} \in B_{2}$ such that $f(a)=b_{1}$ and $f(a)=b_{2}$. Since $f$ is a function, $b_{1}$ and $b_{2}$ must be the same thing; call it $b$, and note that $b \in B_{1} \cap B_{2}$. Therefore, $a \in f^{-1}\left(B_{1} \cap B_{2}\right)$. This shows that the right side is a subset of the left side. Therefore, the two sets are equal.

## 8. (25 pts.) (Answers may be left in terms of factorials.)

(a) How many ways are there to order the letters in the word "MISSISSIPPI" with both P's adjacent (but no other restrictions)?
Treat the two P's as a single item. Count all permutations of the 10 items, and divide by the permutations of the items that must be regarded as identical:

$$
\frac{10!}{1!4!4!1!}
$$

where the denominator factors count permutations of M, I, S, and PP, respectively.
(b) Mrs. Brown's 2nd-grade class contains 11 children. In how many ways can they sit around a round table? (Only the relation of children to their neighbors is relevant, not the relation of children to particular chairs.)
The first child can sit anywhere, and then we count the possible orderings of the others:
(c) The children will line up on a stage to spell out "MISSISSIPPI" with placards. In how many ways can they be assigned their letters?
Method 1: There are 11 candidates to be "M". Then there are $10 \times 9 \times 8 \times 7$ choices to be I's, but we need to divide by 4 ! since all orderings of the 4 chosen are equivalent. Similarly, we then have $6 \times 5 \times 4 \times 3 / 4$ ! ways to choose S's from the remaining kids. The two left over have to be P's, so there are no more choices. Now notice that if we had assigned the letters in a different order, the details would have been different but the final answer must be the same. This leads one to discover a better approach to the problem:
Method 2: Mrs. Brown can assign the letters by writing them down on her class roster. Thus the problem is merely the one of counting the orderings of the letters in "MISSISSIPPI" (this time with no constraint that the P's be together):

$$
\frac{11!}{1!4!4!2!}
$$

9. (10 pts.) Prove that $3 n^{2}-n \in \Theta\left(n^{2}\right)$. (Quote the definitions that show what this statement means, and then show that they are satisfied.)
We need to show that $3 n^{2}-n \in O\left(n^{2}\right)$ and that $n^{2} \in O\left(3 n^{2}-n\right)$. The first of these conditions says that there is a constant $C$ such that $\left|3 n^{2}-n\right| \leq C\left|n^{2}\right|$ at least for sufficiently large $n$. This is clearly true for $C=3$. Going the other way, we must show that there is a $K$ such that $\left|n^{2}\right| \leq K\left|3 n^{2}-n\right|$ for large $n$. Since $n^{2}-n \geq 0$, we have $n^{2} \leq 2 n^{2}-n \leq 3 n^{2}-n$, so the condition holds with $K=1$. (The absolute-value signs were dropped because the expressions are all positive in this case.)
10. (Essay - 20 pts.) Tell what you know about ONE of these obscure or optional topics. (Extra credit for TWO - please indicate which two you want graded!)
(A) Russell's paradox and/or uncomputable functions
(B) Projections and relational databases
(C) Stirling numbers (not "Stirling's formula" for $n$ !)
(D) The Chinese remainder theorem
(E) Equivalence relations, partitions, and the relation between them (This one is not really obscure or optional, so it will be graded fairly stringently.)
