Logic

One reason for studying logic is that we need a better notation than ordinary English for expressing relationships among various assertions or hypothetical states of affairs. A solid grounding in formal logic would make it easier for freshmen to understand the definition of continuity:

> The function f is continuous if, for every x in the domain of f, for every number $\epsilon > 0$ there is a number $\delta > 0$ such that for any number y in the domain, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$.

So, why isn't calculus taught that way? Probably the best reason is that logic is more abstract than the calculus itself, so students would not understand or appreciate it if college mathematics led off with it. You can't understand a solution until you first understand why there is a problem. To appreciate abstraction you first need some painful experience with the more concrete problems that it clarifies.

Quantifiers

$\forall x$ means "For all x":

$$\forall x\,[x < x + x^2 + 1] \qquad \text{(real numbers understood).}$$

$\exists x$ means "There exists an x such that" [or " ... for which"]:

$$\exists x\,[x \text{ is rotten in Denmark}].$$

You can think of *open sentences* such as

$$x < x + x^2 + 1$$

and

$$x \text{ is rotten in Denmark}$$

as being something like *functions* or *formulas* in algebra and calculus, except that the *value* of such a function, when something particular is plugged in in the place of x, is not a number, but rather a *truth value* — either True or False (or either "Yes" or "No").

A *quantifier*, $\forall x$ or $\exists x$, closes off an open sentence and turns it into a genuine sentence, which is either true or false (although we may not know which). They are very much like the definite integral and limit notations in calculus, which turn formulas into numbers:

$$\int_0^1 x^2\,dx \qquad \text{and} \qquad \lim_{x\to 2} x^2$$

are particular numbers, even though the expressions representing them involve a variable, x. In calculus such a variable is often called a "dummy variable"; in logic it's traditionally called a "bound variable" (tied to its quantifier).

The quantified variable stands for objects in some "universe of discourse", which may be stated explicitly —

$$\forall x\,[\text{if } x \text{ is a real number, then } x < x + x^2 + 1]$$

— or understood from context. This is just like the "domain" of a numerical function. (Don't think of "universe" in the astronomical sense.)

Leading universal quantifiers are often omitted when we state "identities" in mathematics:

$$x + y = y + x; \qquad (n+1)! = (n+1)n!$$

If two or more quantifiers of the same type are adjacent, their order doesn't matter:

$$\forall x \forall y\,[x + y < y + x + 1]; \qquad \forall y \forall x\,[x + y < y + x + 1].$$

However, the order of quantifiers of different type is extremely important. (This is the first remark in this lecture of real technical importance.) Consider, for example, the old saying

Behind every successful man there is a woman.

The structure of this proposition (whether or not you believe it to be true or false) is

$$\forall x \exists y\,[\text{if } x \text{ is a man and } x \text{ is successful, then } y \text{ is a woman and } y \text{ is behind } x].$$

But

$$\exists y \forall x\,[\text{if } x \text{ is a man and } x \text{ is successful, then } y \text{ is a woman and } y \text{ is behind } x]$$

says something completely different: There is *one particular woman* who stands behind every man in the world! It is notorious that this point is important for the ϵ and δ in the definition of a limit: You have to know ϵ before you can choose the right δ. There is not (usually) one δ that works for all ϵ's.

If two quantifiers of one type are separated by one (or more) of the other type, then they cannot be reversed:

$$\exists x \forall y \exists z \quad \text{is not equivalent to} \quad \exists z \forall y \exists x;$$

"There is a country where behind every man there is a woman" is not equivalent to "There is a woman such that for every man there a country where she stands behind him."

Propositional Calculus (The Logical Connectives)

Letters $p, q, \ldots$ are used as variables standing for sentences or open sentences. The second major part of logical notation expresses how simple sentences are combined into compound ones.

AND: $p \wedge q$ (same as "&").

OR: $p \vee q$

It is easy to see that these operations are commutative and associative, so we can write things like $p \wedge q \wedge r$. Furthermore, each of them is distributive over the other:

$$p \wedge (q \vee r) \quad \text{is equivalent to} \quad (p \wedge q) \vee (p \wedge r),$$

$$p \vee (q \wedge r) \quad \text{is equivalent to} \quad (p \vee q) \wedge (p \vee r).$$

(We shall prove this with truth tables in a moment.)

NOT: $\neg p$ ($\neg\neg p$ simplifies to p.)

DeMorgan's Laws:

$$\neg(p \wedge q) \quad \text{is equivalent to} \quad \neg p \vee \neg q.$$

$$\neg(p \vee q) \quad \text{is equivalent to} \quad \neg p \wedge \neg q,$$

There are some closely related laws for quantifiers:

$$\neg\forall x\,[\ldots] \quad \text{is equivalent to} \quad \exists x \neg[\ldots],$$

$$\neg\exists x\,[\ldots] \quad \text{is equivalent to} \quad \forall x \neg[\ldots]$$

Note that for a *finite* universe, quantifiers are unnecessary — they can be rewritten in terms of connectives! If the universe has only 3 elements, say a, b, c, then

$$\forall x\, p(x) \quad \text{is equivalent to} \quad p(a) \wedge p(b) \wedge p(c),$$

$$\exists x\, p(x) \quad \text{is equivalent to} \quad p(a) \vee p(b) \vee p(c).$$

Then the action of $\neg$ on quantifiers follows from DeMorgan's laws.

Both of these sets of laws are especially nice when there are negations on both sides:

$$\neg(\neg p \wedge \neg q) \quad \text{simplifies to} \quad p \vee q,$$

$$\neg\forall x\, \neg p(x) \quad \text{simplifies to} \quad \exists x\, p(x),$$

etc.

Truth Tables

It is convenient and standard to let 1 represent "True" or "Yes" and 0 represent "False" or "No".

Each connective can be precisely defined by telling what its truth value is for each possible truth value of its parts.

AND:

$p \backslash q$	0	1
0	0	0
1	0	1

OR:

$p \backslash q$	0	1
0	0	1
1	1	1

From these the truth tables of more complicated sentences can be deduced. [Do for both sides of one of the distributive laws.]

Implication and logical equivalence

We consider two more extremely important connectives (postponing the touchy question of their names):

$\rightarrow$:

$p \backslash q$	0	1
0	1	1
1	0	1

$\longleftrightarrow$:

$p \backslash q$	0	1
0	1	0
1	0	1

Observe:

1. $p \longleftrightarrow q$ says that p and q have the *same truth value* (either both true or both false). Therefore, if we know one, we can conclude the other.
2. $p \longleftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$.
3. $p \rightarrow q$ is intended to symbolize that *if we know that p is true, then we can conclude that q is true.* Note that the bottom line of its truth table guarantees this.
4. The most common English rendering of $p \rightarrow q$ is, "If p, then q."

The connective $\rightarrow$ is slightly subtle conceptually. Right now you may be wanting to ask:

1. What is the justification for the top line of the table? Does it make sense to say, "If $2+2=5$, then e^x is a continuous function," or "If $2+2=5$, then e^x is a discontinuous function."?
2. Does it make sense to say, "If China is in Asia, then e^x is a continuous function," when the two statements obviously have no connection with each other?

3. What is the difference between the single-shafted arrows ($\rightarrow$, $\longleftrightarrow$) and the double-shafted arrows ($\Rightarrow$, $\Longleftrightarrow$)?

All these questions are related.

The answer to 1 is that we want $p \;\rightarrow\; q$ to be meaningful and useful when p and q are *open sentences* — in particular, when they are inside *quantified* sentences:

$$\forall x \left[\text{If } |x| < \frac{\pi}{4}, \text{ then } |\sin x| < \frac{1}{\sqrt{2}}.\right]$$

This is a true and useful theorem. It does what we want of a theorem: Whenever a number is less than $\frac{\pi}{4}$ in magnitude, it enables us to conclude (correctly) that its sine is less than $\frac{1}{\sqrt{2}}$. However, there are other numbers, such as $x = \frac{\pi}{2}$, for which both the hypothesis and the conclusion are false, and there are still other numbers, such as $x = \pi$, for which the hypothesis is false but the conclusion is true. We must demand that these cases be *consistent with the theorem*, and the connective $\rightarrow$ is defined to make this so. If we changed the top line of the truth table, or left it undefined, then the theorem would become false, or indeterminate, for some of these cases. This would make the formulation of mathematical statements very cumbersome.

The resolution of point 2 is similar. Propositional calculus is concerned only with the *truth values* of sentences, not with what they *mean*. There are only 2 truth values, Yes and No. In this sense all true sentences are the same, and all false ones are the same, just as all numbers 9 are the same, regardless of what things you counted to get the number 9. Therefore, to say

If China is in Asia, then e^x is a continuous function.

is no more strange than to say

> The number of planets in the solar system is less than or equal to the number of states in the Union.

There is no scientific law that makes the latter statement true; it is simply a fact.

Now, about those arrows. In algebra, we make *statements about numbers*. In logic, we make *statements about statements*, and this creates some new conceptual complications. In the technical language of logic, the connectives $\rightarrow$ and $\longleftrightarrow$ belong to the *object language*; they are always *part of a statement under discussion*. The symbols $\Rightarrow$ and $\Longleftrightarrow$ belong to the *metalanguage*; they are used *to talk about statements* and their relationships to each other.

$$p \wedge q \;\rightarrow\; q$$

is a statement (or statement framework) that we can write down, contemplate, test with a truth table, etc.

$$p \wedge q \;\Rightarrow\; q$$

is an *operational principle*: it says that if I am given $p \wedge q$ as one of the premises in an argument, then I can validly conclude q. The distinction is much like that between

"The score written on your paper is at least 90" and "You made an A on the test." The statements are somehow equivalent, but they exist on different levels of meaning: one is about ink, the other is about a person.

In the discussion of formal elementary logic in this chapter, where we are not allowed to assume any information except what is stated in the premises, the single-shafted and double-shafted arrows are effectively equivalent (in the same sense as the two statements about a test). (The point here is that we can't assert a single-shafted statement as a theorem in this context unless it is a *tautology*, not merely true because of some extraneous information about the truth of the primitive statements within it.) However, in technical discussions of advanced logical systems or scientific theories with axioms or background information that is not explicitly stated in the premises, the question of whether one statement can be logically derived from another one is not necessarily the same as whether the if–then statement combining them is true according to the truth table for $\rightarrow$. For example, one would probably deny that

$$\text{The planet Mercury is hot} \;\Rightarrow\; \text{China is in Asia},$$

although

$$\text{The planet Mercury is hot} \;\rightarrow\; \text{China is in Asia}$$

is certainly true.

Terminology and language quirks

1. A common logical terminology:

$\rightarrow$	conditional
$\longleftrightarrow$	biconditional
$\Rightarrow$	implication
$\Longleftrightarrow$	equivalence

   However, Grimaldi uses "implication" for "conditional". Don't worry about this.

2. Much more important in practice: If $p \Rightarrow q$, or $\forall x\,[p(x) \rightarrow q(x)]$, then one says

   $$p \text{ is a } \textit{sufficient condition} \text{ for } q$$

   and

   $$q \text{ is a } \textit{necessary condition} \text{ for } p$$

   (that is, if q is false, then p can't be true). Therefore, if p is both necessary and sufficient for q, then $p \Longleftrightarrow q$ or $\forall x\,[p(x) \longleftrightarrow q(x)]$ (and conversely — the "if … then" in this sentence is really a "$\Longleftrightarrow$"!).

3. For $p \rightarrow q$ we have

contrapositive:	$\neg q \Rightarrow \neg p$
converse:	$q \Rightarrow p$
inverse:	$\neg p \Rightarrow \neg q$

The original statement and its contrapositive are logically equivalent. The converse and the inverse are equivalent, because the inverse is the contrapositive of the converse. But the original and the converse are *not* logically equivalent (although they may both be true (or both false) under certain circumstances).

Example of 2 and 3: Recall infinite series, $\sum_{n=0}^{\infty} a_n$. Let

$$p \equiv \quad a_n \to 0 \quad \text{as } n \to \infty,$$

$$q \equiv \quad \text{The series converges.}$$

Then p does *not* imply q, but q *does* imply p (e.g., the harmonic series). Thus

(A) Tending of the terms to 0 is a *necessary* condition for convergence, but not a *sufficient* condition. (In contrast, most of the series convergence theorems state *sufficient* conditions (e.g., the alternating series test or the ratio test).)

(B) $p \to q$ is false for the harmonic series, but its converse, $q \to p$, is true. (Of course, $p \to q$ is true of all series that happen to be convergent.) If we attach a universal quantifier over all series, $a \equiv \{a_n\}$, then the result

$$\forall a\,[p(a) \ \to \ q(a)]$$

is false, but

$$\forall a\,[q(a) \ \to \ p(a)]$$

is true. As a rule of reasoning, $p \Rightarrow q$ is false but $q \Rightarrow p$ is true.

4. It is sometimes said that the English counterpart of $\to$ is IF, but this is not true in the same sense that the English counterpart of $\wedge$ is AND. Notice that $p \to q$ can be expressed as

$$\text{If } p\text{, then } q,$$

but that

$$p \text{ IF } q$$

corresponds instead to $q \Rightarrow p$. However, if we say

$$p \text{ ONLY IF } q,$$

then we do get something that means $p \to q$; it says that if q is false, then p is false, which is the contrapositive of $p \to q$. A consequence of this is that

$$p \text{ IF AND ONLY IF } q$$

is an English way of saying $p \longleftrightarrow q$. In fact, it is the standard way of expressing equivalent conditions (usually $\forall x\,[p(x) \longleftrightarrow q(x)]$ statements) in mathematical English. Often it is "blackboard abbreviated" to "IFF".

5. **Counterfactual conditionals:** Contemplate this sentence:

   > If the Federal Reserve had not lowered the interest rate, we would be in a recession now.

   Is it *automatically true* by the truth table for $\rightarrow$, because the Fed did lower the interest rate? That doesn't seem right.

   Now consider this one:

   > If Napoleon and Julius Caesar had been contemporaries, then Napoleon would be 2000 years old by now.

   Let's come back to these puzzles later, if we have time.