Test A – Solutions

_____ Number: _____

(as on attendance sheets)

Calculators may be used for simple arithmetic operations only!

1. (10 pts.) Show that the number of seconds in 6 weeks is exactly 10!. We have

$$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$$

and also

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$$60 \operatorname{sec/min} \cdot 60 \operatorname{min/hr} \cdot 24 \operatorname{hr/day} \cdot 7 \operatorname{day/wk} \cdot 6 = (10 \cdot 3 \cdot 2) \cdot (5 \cdot 4 \cdot 3) \cdot (8 \cdot 3) \cdot 7 \cdot 6.$$

Combining two 3s into a 9 and reordering, we see that the factors match. *Remark:* This assertion was made without proof on p. 7 of the textbook.

2. (15 pts.) Prove by induction that $\sum_{k=2}^{n} \frac{1}{k^2 - 1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}$ for all integers $n \ge 2$.

Explain the argument in complete sentences! First, we verify that the identity is true for n = 2:

$$\frac{1}{3} = \frac{1}{2^2 - 1} = \frac{3}{4} - \frac{5}{4 \cdot 3} = \frac{9 - 5}{12} = \frac{1}{3}.$$

Now assume the identity for n and consider n+1:

$$\begin{split} \sum_{k=2}^{n+1} \frac{1}{k^2 - 1} &= \sum_{k=2}^n \frac{1}{k^2 - 1} + \frac{1}{(n+1)^2 - 1} \\ &= \frac{3}{4} - \frac{2n+1}{2n(n+1)} + \frac{1}{[(n+1) - 1][(n+1) + 1]} \\ &= \frac{3}{4} - \frac{2n+1}{2n(n+1)} + \frac{1}{n(n+2)} \\ &= \frac{3}{4} + \frac{-(n+2)(2n+1) + 2(n+1)}{2n(n+1)(n+2)} \\ &= \frac{3}{4} + \frac{-2n^2 - 4n - n - 2 + 2n + 2}{2n(n+1)(n+2)} \\ &= \frac{3}{4} + \frac{-2n^2 - 3n}{2n(n+1)(n+2)} \\ &= \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)} \\ &= \frac{3}{4} - \frac{2(n+1) + 1}{2(n+1)[(n+1) + 1]} \end{split}$$

which is what we needed to prove.

In Questions 3 and 4 it is permissible to leave the answers in terms of factorials.

- 3. (28 pts.) Bob, Carol, Ted, and Alice went into Joe's Diner and sat at a table for 6.
 - (a) In how many different ways could these individuals choose their seats?

$$5 \cdot 5 \cdot 4 \cdot 3 = \frac{6!}{2!}$$

(b) The menu lists hamburger, fajitas with either green chili or red chili, hot dog, and Super TacoTM, along with tea, lemonade, and CokeTM. How many different food orders could they put in, if each person orders one main dish and one drink? (The server simply brings the tray of food to the table, rather than distributing each food to the person who wants it.)

Note that there are 5 main dishes; there is no need to treat the two kinds of fajitas in any special way. We need to multiply the number of ways to put 4 things into 5 boxes by the number of ways to put 4 things into 3 boxes. Remembering that the number of dividers is one less than the number of boxes, we get

$$\binom{4+4}{4}\binom{4+2}{4} = \frac{8!}{4!4!} \cdot \frac{6!}{4!2!} = \frac{8!6!}{4!^32!}$$

(c) How did the answer to (b) change when the server said, "I'm sorry, we have only 2 Super Tacos left today"?

Add the numbers for 0, 1, and 2 taco orders; in each case the remaining people must go into 4 boxes. The beverage factor is unchanged.

$$\binom{6}{4} \left[\binom{4+3}{4} + \binom{3+3}{3} + \binom{2+3}{2} \right] = \frac{6!}{4! \, 2!} \left[\frac{7!}{4! \, 3!} + \frac{6!}{3! \, 3!} + \frac{5!}{2! \, 3!} \right].$$

(d) How does the answer to (b) change if the server distributes the main dishes to individuals but still leaves the drinks on one tray?

Again the beverage factor is unchanged. The other factor simplifies:

$$\binom{6}{4} \cdot 5^4.$$

- 4. (12 pts.) Consider the expansion of $(u + v + w + x + y + z)^4$. (Don't write it out!)
 - (a) After the expression is simplified (by the commutative rule), how many terms will there be?
- Put 4 things (factors) into 6 boxes (variables):

$$\binom{4+6-1}{4} = \frac{9!}{4!\,5!}\,.$$

(b) What will be the coefficient of xy^2z ? The multinomial coefficient

$$\frac{4!}{0!^3 1!^2 2!} = 12$$

5. (20 pts.) Consider the symbolic proposition $[p \to (q \lor r)] \longleftrightarrow [(p \land \neg q) \to r]$. (a) Construct a truth table for the proposition.

p	q	r	$q \vee r$	$p \wedge \neg q$	A	B	$A \longleftrightarrow B$
0	0	0	0	0	1	1	1
0	0	1	1	0	1	1	1
0	1	0	1	0	1	1	1
0	1	1	1	0	1	1	1
1	0	0	0	1	0	0	1
1	0	1	1	1	1	1	1
1	1	0	1	0	1	1	1
1	1	1	1	0	1	1	1

(b) Is the proposition a tautology, a contradiction, or neither?

A tautology (because the last column is all 1s).

- 6. (15 pts.) Express these (paraphrased) excerpts from the Instructions for Form 1040 in logical notation. (Replace by quantifiers and connectives all "logical" words such as "if", "every", "or", and "provided that".)
 - (a) Every U.S. citizen, and any person who earns income in the United States, must file a tax return.

 $\forall x [(x \text{ is a US citizen}) \lor (x \text{ earns income in the US}) \rightarrow (x \text{ must file a return})].$

(b) A taxpayer may claim his daughter as a dependent, provided that she is not married and filing a joint return, if he provides more than half of her support, or if he and another person together provide more than half of her support and he paid over 10% of her support.

 $\forall x \forall d [(d \text{ is a daughter of } x) \land \neg(d \text{ is married} \land d \text{ files a joint return})$

 $\wedge \{(x \text{ provides more than half of } d's \text{ support})\}$

 $\vee [\exists y(x \text{ and } y \text{ provide more than half of } d's \text{ support}) \land x \text{ provides over } 10\% \text{ of } d's \text{ support})]\}$

 \rightarrow (x may claim d as a dependent)].

(It is also legitimate to write a chain of conditionals, schematically

daughter \rightarrow {(not married ...) \rightarrow [(support clauses) \rightarrow dependent]},

for example.)