Math. 302 (Fulling) 8 March 2002

Test B – Solutions

Name: ________________________ Number: ____________
(as on attendance sheets)

Calculators may be used for simple arithmetic operations only!

1. *(12 pts.)* A and B are sets with $|B| = 5$. There are exactly $32768 = 2^{15}$ relations from A to B.

   (a) What is $|A \times B|$ (the cardinality of the Cartesian product)?

$15 = \log_2(2^{15})$.

   (b) What is $|A|$?

$\dfrac{|A \times B|}{|B|} = 3$.

2. *(16 pts.)* One of these formulas is valid, the other is not. Prove the correct one and provide a counterexample for the other. *Suggestion:* For the counterexample, let $p(x,y)$ be an inequality in the universe of real numbers.

   (a) $\forall x\, \exists y\, p(x,y) \;\rightarrow\; \exists y\, \forall x\, p(x,y)$

Not valid. Let $p(x,y)$ be $x < y$. Then the hypothesis is true, because, given x, y could be $x+1$. However, the conclusion is false, because, whatever y is, one can find an x that is larger, say $x = y+1$.

   (b) $\exists y\, \forall x\, p(x,y) \;\rightarrow\; \forall x\, \exists y\, p(x,y)$

Valid. Here is a formal deduction:

(1) $\exists y\, \forall x\, p(x,y)$ (hypothesis)
(2) Let c be such a y: $\forall x\, p(x,c)$ (existential specification)
(3) Let d be arbitrary: $p(d,c)$ (universal specification)
(4) $\exists y\, p(d,y)$ (existential generalization)
(5) Since d was arbitrary, $\forall x\, \exists y\, p(x,y)$ (universal generalization)
(6) Therefore, $\exists y\, \forall x\, p(x,y) \;\rightarrow\; \forall x\, \exists y\, p(x,y)$ (because we've shown that (1) implies (5))

Remark: Let's see why this type of argument fails in case (a):

(1) $\forall x\, \exists y\, p(x,y)$ (hypothesis)
(2) Let d be arbitrary: $\exists y\, p(d,y)$ (universal specification)
(3) Let c be such a y: $p(d,c)$ (existential specification)
(4) $\forall x\, p(x,c)$ (universal generalization??) **This is wrong, because in (3) c depends on x!** (See W. V. Quine, *Methods of Logic*, for instructions on how to use "flagged variables" to avoid this kind of error.)

3. *(21 pts.)* Recall that a pair of dice consists of two cubes, the sides of each of which are labeled by the numbers 1 through 6. When the dice are thown, what is the probablity of obtaining each of these results? (Leave the answer as a fraction in lowest terms, not a decimal.)

   (a) a pair (i.e, the same number on each die)

Note first that there are $6^2 = 36$ total possibilities, since the 6 possibilities for each die are independent of those for the other die. Of these, 6 outcomes are pairs, so the probability of a pair is $\frac{6}{36} = \frac{1}{6}$.

(b) a total of 4

There are 3 ways to get a 4: $1+3$, $2+2$, or $3+1$. So the probability is $\frac{3}{36}=\frac{1}{12}$.

(c) either a total of 4, or a pair

All of the cases found in (a) and (b) count, but the case $2+2$ must not be counted twice. So the total number of good outcomes is $6+3-1=8$, for a probability of $\frac{8}{36}=\frac{2}{9}$. This is an instance of the principle

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

4. *(16 pts.)* Establish the validity of the argument

$$\{(p \to q) \wedge [(q \wedge r) \to s] \wedge r\} \Rightarrow (p \to s).$$

(Write a formal deduction, not a truth table.)

(1) $\{(p \to q) \wedge [(q \wedge r) \to s] \wedge r\}$ (hypothesis)
(2) p (hypothesis of the conclusion)
(3) $(p \to q)$ (part of (1))
(4) q (modus ponens, (2) and (3))
(5) r (part of (1))
(6) $q \wedge r$ (from (4) and (5))
(7) $(q \wedge r) \to s$ (part of (1))
(8) s (modus ponens, (6) and (7))
(9) $p \to s$ (because we have shown that (2) implies (8))

This shows that the hypothesis (1) logically implies the conclusion (9).

5. *(15 pts.)*

(a) Prove that $A \cup (A \cap B) = A \cap (A \cup B)$, or give a counterexample.

Method 1: Apply the distributive law for sets:

$$A \cup (A \cap B) = (A \cup A) \cap (A \cup B) = A \cap (A \cup B).$$

(Or apply the other distributive law from the other direction.)

Method 2: Appeal to the absorption law (part (b)) to show that both sides are equal to A.

Method 3: An "element proof":

$$x \in A \cup (A \cap B) \Rightarrow (x \in A) \vee (x \in A \cap B).$$

But $x \in A \cap B \to x \in A$. So in either case of "$\vee$" we have $x \in A$. And if $x \in A$, then also $x \in A \cup B$. And therefore $x \in A \cap (A \cup B)$. So far we have shown that

$$A \cup (A \cap B) \subseteq A \cap (A \cup B).$$

Conversely, assume that $x \in A \cap (A \cup B)$. Then $x \in A$ and $x \in A \cup B$. But if $x \in A$, then certainly $x \in A \cup (A \cap B)$, and we have now shown that

$$A \cap (A \cup B) \subseteq A \cup (A \cap B).$$

So the two sets are equal.

Method 4: Start from the logical absorption law, $p \wedge (p \vee q) \longleftrightarrow p \vee (p \wedge q)$, or the logical distributive law, $p \wedge (q \vee r) \longleftrightarrow (p \wedge q) \vee (p \wedge r)$; let p be $x \in A$, etc., and apply the definitions of the set-theory operations.

(b) Can you simplify either side of the equation to something else?

Yes, both sides simplify to A. This is called the "absorption law" for sets.

6. *(20 pts.)* Prove by induction that (for $n = 0, 1, \dots$)

$$\frac{d^n}{dx^n}[f(x)g(x)] = \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(x) g^{(n-j)}(x).$$

(Here $f^{(j)}$ is the j th derivative of f, etc.) *Hints:* Use $\binom{n-1}{j-1} + \binom{n-1}{j} = \binom{n}{j}$. Start by showing that the case $n = 1$ is a well known fact of calculus.

The case $n = 1$ is the product rule, $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$. The case $n = 0$ is trivial: $f(x)g(x) = f(x)g(x)$. We need to see what happens when we apply the product rule repeatedly. Let's write out the formula for the case $n - 1$:

$$\frac{d^{n-1}}{dx^{n-1}}(fg) = \sum_{j=0}^{n-1} \binom{n-1}{j} f^{(j)} g^{(n-1-j)}.$$

Then

$$\begin{aligned}
\frac{d^n}{dx^n}(fg) &= \frac{d}{dx}(fg)^{(n-1)} \\
&= \sum_{j=0}^{n-1} \binom{n-1}{j} \left[f^{(j+1)} g^{(n-1-j)} + f^{(j)} g^{(n-j)} \right] \\
&= \sum_{j=1}^{n} \binom{n-1}{j-1} f^{(j)} g^{(n-j)} + \sum_{j=0}^{n-1} \binom{n-1}{j} f^{(j)} g^{(n-j)} \\
&= \binom{n-1}{0} f^{(0)} g^{(n)} + \binom{n-1}{n-1} f^{(n)} g^{(0)} + \sum_{j=1}^{n-1} \left[\binom{n-1}{j-1} + \binom{n-1}{j} \right] f^{(j)} g^{(n-j)} \\
&= f^{(0)} g^{(n)} + f^{(n)} g^{(0)} + \sum_{j=1}^{n-1} \binom{n}{j} f^{(j)} g^{(n-j)} \\
&= \sum_{j=0}^{n} \binom{n}{j} f^{(j)} g^{(n-j)}.
\end{aligned}$$

Remark: The hint is the identity on which Pascal's triangle is based. So the theorem says that repeated differentiation involves exactly the same combinatorics as the binomial theorem.