## Test C - Solutions

Name: $\qquad$ Number: $\qquad$
(as on attendance sheets)

## Calculators may be used for simple arithmetic operations only!

1. (25 pts.) The matrix $M=\left(\begin{array}{lll}1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right)$ represents a relation $\mathcal{R}$ on the set $C=\{1,2,3\}$.
(a) Draw a directed graph representing $\mathcal{R}$.

(b) Calculate the matrix representing $\mathcal{R} \circ \mathcal{R}$.

The matrix is $M^{2}$, calculated with the "logical" definition of addition:

$$
M^{2}=\left(\begin{array}{lll}
1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)\left(\begin{array}{lll}
1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)=\left(\begin{array}{lll}
1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right) .
$$

Thus $M^{2}=M$ in this problem. (The reason for this, in terms of the graph, is that the only two-step paths are those that have a loop at either the beginning or the end, and in all cases such a loop is available. Thus the two-step paths correspond precisely to the one-step paths (including the loops themselves).)
(c) Show that $\mathcal{R}$ is an (inclusive partial) ordering (i.e., $\mathcal{R}$ is reflexive, antisymmetric, and transitive). Explain each property clearly!
Each of these properties can be observed from either the matrix or the graph.
$\mathcal{R}$ is reflexive because every element on the main diagonal of the matrix is a 1 , or because every vertex (labeled point) in the graph has a loop attached.
$\mathcal{R}$ is antisymmetric because no two distinct vertices are connected by arrows in both directions. In terms of the matrix, this condition is that an off-diagonal matrix element and its reflection through the main diagonal are never both 1 (although they may be both 0 , and are in one case in this example). A more algebraic statement of this condition is that $M \cap M^{\mathrm{t}} \leq I$, where $I$ is the matrix with all 1 s on the diagonal and all 0 s off the diagonal, and

$$
M \cap M^{\mathrm{t}}=\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)
$$

is formed by multiplying each matrix element in $M$ by its reflection (the element symmetrically located opposite it through the main diagonal). (In the present case, this matrix turns out to be actually equal to $I$.)
$\mathcal{R}$ is transitive because $M^{2} \leq M$, as we found in (b). The graphical reason for this has, in effect, already been discussed in the commentary on (b). Transitivity in this case is rather trivial, because there are no nontrivial two-step paths: at least one of the two steps is always a loop, leaving the other step marking the result of the double step.
2. (25 pts.) Let $A=\{$ Texas, Oklahoma, Kansas $\}$ and $B=\{a, b, c, d\}$. Count these:
(a) Functions from $A$ to $B$.
$4^{3}=64$.
(b) Injective (one-to-one) functions from $A$ to $B$.
$P(4,3)=4 \times 3 \times 2=24$.
(c) Surjective (onto) functions from $A$ to $B$.

0 . (There aren't enough elements in $A$ to use up $B$.)
(d) Relations from $B$ to $A$.
$2^{4 \times 3}=2^{12}=4096$.
(e) Nonconstant functions from $B$ to $A$.

There are 3 constant functions, so the nonconstant functions are the others:

$$
3^{4}-3=81-3(\text { or } 3(27-1))=78 .
$$

3. (13 pts.) Professor Emptyne wants to give his class a test question involving an $N \times N$ matrix (a table of numbers with $N$ rows and $N$ columns). To keep the arithmetic easy, each element should be a one-digit integer. (Negative numbers are allowed.) To avoid numerical coincidences, he would like all the numbers to be distinct. How big can $N$ be?
There are 19 one-digit integers: $\{-9, \ldots,-1,0,1, \ldots, 9\}$. The number of places in the matrix is $N^{2}$. When $N=5, N^{2}=25$ - which is too big, by the pigeonhole principle. When $N=4$, $N^{2}=16$, which will work. So 4 is the largest $N$ allowed.
4. (24 pts.) Let $f(n)=500 n^{2}+3 n, g(n)=n, h(n)=n \ln n, k(n)=3 n \ln (3 n)$. Decide whether each of these statements is true or false.
(a) $f \in O\left(n^{2}\right)$

TRUE.
(b) $\quad g \in \Theta\left(n^{2}\right)$

FALSE.
(c) $\quad g \in O(f)-\Theta(f)$

## TRUE.

(d) $\quad h \in \Omega(g)$

TRUE.
(e) $\quad h \in \Theta(k)$

TRUE. Note that $k(n)=3 h(n)+3 n \ln 3$, and the last term has slower growth than $h$.
(f) $\quad k \in O(f)$

TRUE.
5. (13 pts.) Prove that if a set $A$ has $n$ elements $(|A|=n)$, then there are $n^{n(n+1) / 2}$ commutative closed binary operations on $A$. (Partial credit if you do this only for a particular value of $n$, such as $n=3$.)
We must assign an element in $A$ to each element (pair) $(a, b)$ in $A \times A$, with the sole restriction that $(b, a)$ gets assigned the same result as $(a, b)$. The number of such operations is $n^{p}$, where $p$ is the number of unordered pairs of elements of $A$. This $(p)$ is the same as the number of distinct elements in a symmetric $n \times n$ matrix; in other words, the number of elements above or on the main diagonal of the matrix, the elements below the diagonal being ignored as redundant. There are (at least) two ways to see that $p=\frac{1}{2} n(n+1)$ : (1) We need to count the $n$ on-diagonal elements plus half of the $n^{2}-n$ off-diagonal elements. That makes

$$
n+\frac{n^{2}-n}{2}=\frac{n^{2}+n}{2}=\frac{1}{2} n(n+1)
$$

(2) Counting from the diagonal up toward the top right corner, we have

$$
n+(n-1)+\cdots+1=\sum_{j=1}^{n} j=\frac{1}{2} n(n+1)
$$

by the most famous of all mathematical induction problems.

