Math. 302 (Fulling) 14 December 2004

Final Examination – Solutions

Calculators may be used for simple arithmetic operations only!

1. *(12 pts.)* For each of these "divide and conquer" recursions, either find an asymptotic estimate on $T(n)$ using the master theorem, or explain why the master theorem does not apply.

   (a) $T(n) = 2T\left(\frac{n}{2}\right) + 50n$.

Here $a = b$, so $\log_b a = 1$. Since $s = 50n \in \Theta(n^1)$, we are in Case 2, with the result $T \in \Theta(n \lg n)$.

   (b) $T(n) = 4T\left(\frac{n}{2}\right) + n^2 \log n$.

Here $\log_b a = \log_2 4 = 2$. Cases 2 and 1 do not apply, because $s = n^2 \log n$ grows faster than n^2 . But we are not in Case 3 either, because s does not grow as fast as $n^{2+\epsilon}$ for any ϵ . Therefore, the theorem does not apply.

2. *(25 pts.)*

   (a) Give an "element proof" of the set-theoretic De Morgan law, $\overline{A \cup B} = \overline{A} \cap \overline{B}$. (In other words, use the *definitions* of the set-theory relations and operations together with the appropriate *logical* De Morgan law.)

$$
\begin{aligned}
x \in \overline{A \cup B} &\iff \neg(x \in A \cup B) \\
&\iff \neg[x \in A \vee x \in B] \\
&\iff \neg(x \in A) \wedge \neg(x \in B) \\
&\iff x \in \overline{A} \wedge x \in \overline{B} \\
&\iff x \in \overline{A} \cap \overline{B}.
\end{aligned}
$$

   (b) Prove by mathematical induction [and (a)] the generalized De Morgan law,

$$\overline{\bigcup_{n=1}^{N} A_n} = \bigcap_{n=1}^{N} \overline{A_n} \qquad (N \geq 2).$$

Base: Part (a) is the case $N = 2$.
Induction: Assume the law is known for the case $N - 1$.

$$
\begin{aligned}
\overline{\bigcup_{n=1}^{N} A_n} &= \overline{\bigcup_{n=1}^{N-1} A_n} \cap \overline{A_N} \quad \text{(by (a))} \\
&= \bigcap_{n=1}^{N-1} \overline{A_n} \cap \overline{A_N} \quad \text{(by inductive hypothesis)} \\
&= \bigcap_{n=1}^{N} \overline{A_n} \,.
\end{aligned}
$$

3. *(13 pts.)* Find a recursion relation (with initial conditions) for the number of (decimal) digit strings of length n that do not contain a pair of consecutive zeros.

Clearly $a_0 = 1$ and $a_1 = 10$, since there is not yet any possibility of repetition of digits. To get an allowed string of length n we can either append something nonzero to an allowed string of length $n-1$ ($9a_{n-1}$ possibilities) or append a 0 to an allowed string that ends in a nonzero ($9a_{n-2}$ possibilities, since the nonzero digit could be added to any allowed string of length $n-2$). Thus

$$a_n = 9a_{n-1} + 9a_{n-2}\,, \qquad a_0 = 1, \quad a_1 = 10.$$

Check: Our formula gives $a_2 = 99$ and $a_3 = 981$. These are correct: For $n = 2$ all 100 strings are allowed except the double zero. For $n = 3$ there are 1000 strings, of which one is a triple zero and $2 \cdot 9 = 18$ have 00 either preceded or followed by a nonzero; that leaves 981.

4. *(30 pts.)* *[Leave answers in terms of factorials and powers.]* Because Prof. Lucas will be snowbound in Labrador for all of final exam week, each of the 20 students in his Math. 320 class will need to take the final with one of the other 3 sections of the course. The students get to choose which class to go to.

   (a) How many ways can the students make those choices (the students being distinguishable)?

3^{20}. (Each student independently has 3 choices.)

   (b) How many patterns of choice are possible if the students are regarded as indistiguishable? (An example of a pattern is: 10 students went to Prof. Smith's exam, 6 to Prof. Bernt's, and 4 to Prof. Woodcock's.)

$\binom{20+(3-1)}{20} = \frac{22!}{2!\,20!}$. (Putting 20 indistinguishable things into 3 distinguishable boxes is the same as separating the 20 things by 2 indistinguishable dividers.)

   (c) What is the coefficient of $x^{10}y^6z^4$ in $(x+y+z)^{20}$?

$\frac{20!}{10!\,6!\,4!}$ (multinomial coefficient).

   (d) Is there any connection between part (c) of this question and parts (a) and (b)? (Explain.)

The answer to (b) is the same as the number of distinct monomial terms, (coefficient) $\times\, x^a y^b z^c$, in $(x+y+z)^{20}$. The sum of *all* the corresponding multinomial coefficients is 3^{20}, the answer to (a); this is a case of the trinomial generalization of the binomial theorem (Pascal's pyramid). Putting it the other way around, the answer to (c) is the number of instances of the example pattern in (b) when names are reattached to the students.

5. *(35 pts.)* Solve these recursion relations, and give one of them its famous name.

   (a) $a_n = a_{n-1} + a_{n-2}$. (Find the general solution.)

This is the Fibonacci recursion. Trying $a_n = r^n$, we get the equation $r^2 - r - 1 = 0$. Its solutions are

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}\,.$$

So the general solution of the recursion is

$$a_n = C_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + C_2 \left(\frac{1-\sqrt{5}}{2} \right)^n .$$

Remark: With $C_1 = -C_2 = \frac{1}{\sqrt{5}}$ we get the familiar Fibonacci numbers, $0\,,\ 1\,,\ 1\,,\ 2\,,\ 3\,,\ 5\,,\ \ldots.$ See Rosen, p. 416.

(b) $a_{n+2} - 9a_n = 5 \cdot 2^n, \qquad a_0 = 1, \quad a_1 = 0\,.$

First find a solution of the homogeneous relation, $a_{n+2} - 9a_n = 0$, of the form $a_n^{\mathrm{h}} = r^n$. We are led to $r^2 - 9 = 0$, or $r = \pm 3$. Thus $a_n^{\mathrm{h}} = C_1 3^n + C_2(-3)^n$.

Now find a particular solution of the nonhomogeneous equation in the form $a_n^{\mathrm{p}} = A \cdot 2^n$. We get $A \cdot 2^{n+2} - 9A \cdot 2^n = 5 \cdot 2^n$, or $(4-9)A = 5$, whence $A = -1$. So the general solution of the nonhomogeneous equation is

$$a_n^{\mathrm{h}} + a_n^{\mathrm{p}} = C_1 3^n + C_2(-3)^n - 2^n.$$

Finally, we need to match the initial data.

$$1 = C_1 + C_2 - 1,$$
$$0 = 3C_1 - 3C_2 - 2.$$

The solution is $C_1 = \frac{4}{3}$, $C_2 = \frac{2}{3}$. So

$$a_n = \frac{4}{3} 3^n + \frac{2}{3}(-3)^n - 2^n = 4 \cdot 3^{n-1} - 2(-3)^{n-1} - 2^n.$$

6. *(14 pts.)*

(a) Rewrite in good English:

$$\forall x \in \mathbf{R}\, \forall y \in \mathbf{R}\ \big[\big((x<0) \wedge (y<0)\big) \ \rightarrow\ xy > 0\big]\,.$$

The product of two negative real numbers is always positive. *(or other words to the same effect)*

(b) Rewrite in logical notation: [Interpret as a *general statement* about numbers.]

A negative real number does not have a square root that is a real number.

$$\forall x \in \mathbf{R}[x < 0 \ \rightarrow\ \neg \exists y \in \mathbf{R}\,(x = y^2)].$$

7. *(20 pts.)* Let F be this set of functions:

$$\left\{ e^n + \ln n\,,\ 2n + \ln n\,,\ \sqrt{n^2+1}\,,\ \frac{e^{2n}+1}{e^n}\,,\ e^{2n}\,,\ \frac{n^2+1}{\log n} \right\}$$

Let $\mathcal{R}$ be the equivalence relation defined on F by

$$(f,g) \in \mathcal{R} \iff f \in \Theta(g)\,.$$

(a) What is the partition of F induced by $\mathcal{R}$? [Answer should be a *list of subsets* (*"equivalence classes" or "cells") of* F.]

Note that the third function is in $\Theta(n)$ and the fourth one is in $\Theta(e^n)$. The leading behavior of the other functions is more obvious, and two of them overlap with these two. So we have 4 cells:

$$\left\{ e^n + \ln n\,,\ \frac{e^{2n}+1}{e^n} \right\},\quad \left\{ 2n + \ln n\,,\ \sqrt{n^2+1} \right\},\quad \left\{ e^{2n} \right\},\quad \left\{ \frac{n^2+1}{\log n} \right\}.$$

(b) Take one representative from each cell of this partition and order these representatives from the slowest to the fastest (in terms of growth at infinity).

$$2n + \ln n \prec \frac{n^2+1}{\log n} \prec e^n + \ln n \prec e^{2n}.$$

Remark: If the question read "Take a maximally simple representative from each Θ class ... ", the answer would be

$$n \prec \frac{n^2}{\log n} \prec e^n \prec e^{2n}.$$

8. *(15 pts.)* Suppose that $d_{n+3} = d_{n+1} + d_n$ for $n \geq 0$ and that $d_0 = 1$, $d_1 = 0$, $d_2 = 1$. Show that

$$\sum_{n=0}^{\infty} d_n x^n = \frac{1}{1 - x^2 - x^3}.$$

(Then stop! You are **not** expected to solve the recursion.)

Define $f(x) = \sum_{n=0}^{\infty} d_n x^n$. Multiply the recursion by x^{n+3} and sum:

$$\sum_{n=0}^{\infty} d_{n+3} x^{n+3} = \sum_{n=0}^{\infty} d_{n+1} x^{n+3} + \sum_{n=0}^{\infty} d_n x^{n+3}.$$

(If you multiply just by x^n, it still works out but the algebra is slightly messier.) Shift indices:

$$\sum_{n=3}^{\infty} d_n x^n = \sum_{n=1}^{\infty} d_n x^{n+2} + \sum_{n=0}^{\infty} d_n x^{n+3}.$$

That is,

$$f(x) - d_0 - d_1 x - d_2 x^2 = x^2[f(x) - d_0] + x^3 f(x).$$

Use the given initial values and solve for $f(x)$:

$$f(x)(1 - x^2 - x^3) = (1 + x^2) - x^2 = 1,$$

so

$$f(x) = \frac{1}{1 - x^2 - x^3}.$$

9. *(16 pts.)* Consider these propositions:

p : Grizzly bears have been seen in the area.

q : Hiking on the trail is safe.

r : Berries are ripe along the trail.

(a) Express in logical notation:

(i) If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been seen in the area.

$$r \rightarrow (q \longleftrightarrow \neg p).$$

(ii) It is not safe to hike on the trail, but grizzly bears have not been seen in the area and the berries along the trail are ripe.

$$\neg q \wedge \neg p \wedge r.$$

(b) Determine (by a truth table or a shortcut argument) whether propositions (i) and (ii) can be true simultaneously.

Method 1: If r is false, then (ii) is false. If r is true, then according to (i), q and p have opposite truth values, which contradicts (ii). Therefore, (i) and (ii) are incompatible.

Method 2: Set up a truth table for (i) $\wedge$ (ii). You will get "F" under the " $\wedge$ " for all eight lines. (If you use " $\iff$ " instead of " $\wedge$ " there will be a "T" line, but it corresponds to a case (namely, $q \wedge p \wedge r$) where both sides are *false*, so the correct conclusion still is that the two sides are not true simultaneously.)

Method 3: Only one of the 8 cases is allowed by (ii), and it makes (i) false.

10. *(20 pts.)* Do **ONE** of these [(A) or (B); note that each has mandatory parts (a), (b), etc.] This time you may earn **extra credit by doing both**.

(A)

(a) List all the nonnegative integers less than 35 that are congruent to 0 modulo 5, and determine their residues modulo 7. (Example: $[10]_5 = 0$ and then $[10]_7 = 3$.)

Here is the list of residues:

$$[0]_7 = 0, \quad [5]_7 = 5, \quad [10]_7 = 3, \quad [15]_7 = 1, \quad [20]_7 = 6. \quad [25]_7 = 4, \quad [30]_7 = 2.$$

(b) Use the Chinese remainder theorem (and (a)) to determine $28 \times 13 + 11$ modulo 35. (Don't write down any number larger than 34, and don't write down any number larger than 6 until the last step!) Notation: Represent each n by its residue pair $([n]_5, [n]_7)$.

Translate into residues and do the arithmetic, immediately throwing away multiples of 5 or 7:

$$(3,0) \cdot (3,6) + (1,4) = (4,0) + (1,4) = (0,4).$$

From the list in (a), the number is 25.

Check: $28 \cdot 13 + 11 = 375 = 10 \cdot 35 + 25$.

(B) A relation R is represented by the matrix

$$M = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Determine (with some indication of your reasoning) whether R is

(a) reflexive.

No. (There is a 0 on the diagonal.)

(b) symmetric.

No. (For example, $M_{12} = 1$ but $M_{21} = 0$.)

(c) transitive.

No. With some work, you find that the Boolean square of M is the 4×4 matrix consisting entirely of ones. Therefore, $M^2 \leq M$ is *false*; the transitivity test fails. (For example, $(2,1) \notin R$, but $(2,1) \in R^2$ because $(2,4) \in R$ and $(4,1) \in R$.)