Test A – Solutions

Name: ________________________________

Calculators may be used for simple arithmetic operations only!

1. (10 pts.) Find a parametric representation of the plane through the points \((1, 3, 9)\), \((2, 2, 2)\), and \((0, 2, 5)\).

First find two vectors parallel to the plane:

\[
\vec{u}_1 = (1, 3, 9) - (2, 2, 2) = (-1, 1, 7), \quad \vec{u}_2 = (0, 2, 5) - (2, 2, 2) = (-2, 0, 3).
\]

Then the general point on the plane is

\[
\vec{r} = (2, 2, 2) + s(-1, 1, 7) + t(-2, 0, 3).
\]

(This is one representation of the plane among many possible ones.)

2. (18 pts.) Solve these systems (introducing parameters when the answer is not unique).

(a) \[
\begin{pmatrix}
3 & 8 \\
1 & 3 \\
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
6 \\
-1
\end{pmatrix}.
\]

Reduce the augmented matrix:

\[
\begin{pmatrix}
3 & 5 & 6 \\
1 & 3 & -1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 3 & -1 \\
3 & 8 & 6
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 3 & -1 \\
0 & -1 & 9
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 26 \\
0 & 1 & -9
\end{pmatrix}.
\]

Therefore, \(x = 26\) and \(y = -9\) is the only solution.

(b) \[
\begin{pmatrix}
3 & 9 \\
1 & 3
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

I will not bother to augment the matrix, since the last column will always be zeros.

\[
\begin{pmatrix}
3 & 9 \\
1 & 3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 3 \\
1 & 3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 3 \\
0 & 0
\end{pmatrix}.
\]

Therefore, the system is equivalent to \(x + 3y = 0\). Thus, if \(y = s\), then \(x = -3s\). The general solution can be written

\[
\vec{r} = s\begin{pmatrix}
-3 \\
1
\end{pmatrix}.
\]
3. (15 pts.) Construct the best affine approximation (also known as the first-order approximation) to \( T(x, y) = \left( \frac{x^3 + 4y^2}{\sqrt{x - y^2}} \right) \) in the neighborhood of the point \( \mathbf{r}_0 = \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) = \left( \begin{array}{c} 5 \\ 1 \end{array} \right) \).

The Jacobian matrix is

\[
J_{\mathbf{r}_0} = \left( \begin{array}{cc} \frac{\partial T_1}{\partial x} & \frac{\partial T_1}{\partial y} \\ \frac{\partial T_2}{\partial x} & \frac{\partial T_2}{\partial y} \end{array} \right) = \left( \begin{array}{cc} \frac{3x^2}{2\sqrt{x - y^2}} & \frac{8y - y}{\sqrt{x - y^2}} \\ \frac{1}{2\sqrt{x - y^2}} & \frac{-y}{\sqrt{x - y^2}} \end{array} \right) \bigg|_{x_0, y_0} = \left( \begin{array}{cc} 75 & 8 \\ \frac{1}{4} & -\frac{1}{2} \end{array} \right).
\]

The approximation then is, in matrix form,

\[
T(x, y) \approx T(x_0, y_0) + J_{\mathbf{r}_0}(\mathbf{r} - \mathbf{r}_0) = \left( \begin{array}{c} 129 \\ 2 \end{array} \right) + \left( \begin{array}{cc} 75 & 8 \\ \frac{1}{4} & -\frac{1}{2} \end{array} \right) \left( \begin{array}{c} x - 5 \\ y - 1 \end{array} \right).
\]

There are many acceptable variations on the notation. In such a problem it is better to leave the answer as a function of \( x - x_0 \) and \( y - y_0 \) instead of combining the \( \mathbf{r}_0 \) terms with the constant term.

4. (16 pts.) A curve \( C \) in two-dimensional space is specified by the parametric equations \( x = t^2, \ y = \sin t \).

(a) Find the tangent vector to \( C \) at the point where \( t = \pi \).

\[
\mathbf{T} = \left( \begin{array}{c} \frac{dx}{dt} \\ \frac{dy}{dt} \end{array} \right) = \left( \begin{array}{c} 2t \\ \cos t \end{array} \right) \bigg|_{t = \pi} = \left( \begin{array}{c} 2\pi \\ -1 \end{array} \right).
\]

(b) Find the directional derivative of \( f(x, y) = x + e^y \) at that point, in the direction of the curve.

We need to normalize the tangent vector:

\[
\hat{T} = \frac{\mathbf{T}}{\sqrt{4\pi^2 + 1}}.
\]

\[
\frac{df}{dT} = \nabla f \cdot \hat{T} = (1 \ e^y)|_{t=\pi} \left( \begin{array}{c} 2\pi \\ -1 \end{array} \right) \frac{1}{\sqrt{4\pi^2 + 1}}.
\]

At the point in question, \( y = 0 \) (and \( x = \pi^2 \), although we don’t need that). Therefore,

\[
\frac{df}{dT} = (1 \ 1) \left( \begin{array}{c} 2\pi \\ -1 \end{array} \right) \frac{1}{\sqrt{4\pi^2 + 1}} = \frac{2\pi - 1}{\sqrt{4\pi^2 + 1}}.
\]
5. (9 pts.) When $u$ is a function of a (real) variable $x$, define $A(u)$ to be a new function of $x$ given by the formula $A(u)(x) = u(x) + \int_0^x t^2 u(t) \, dt$. Prove or disprove that $A$ is a linear operator.

Since each term involves $u$ to the first power, we suspect that $A$ is linear. Let us test it:

$$A(ru_1 + u_2)(x) = (ru_1 + u_2)(x) + \int_0^x t^2 (ru_1 + u_2)(t) \, dt$$

$$= ru_1(x) + u_2(x) + r \int_0^x t^2 u_1(t) \, dt + \int_0^x u_2(t) \, dt = [rA(u_1) + A(u_2)](x).$$

6. (12 pts.) Do ONE of these [(A) or (B)]. (Up to 6 points extra credit for doing the other one.)

(A) Producing a refrigerator requires 0.1 ton of steel and 0.2 ton of plastic. Producing an airplane requires 5 tons of steel and 2 tons of plastic. Producing a ton of steel consumes 3 tons of coal and 10 barrels of water. Producing a ton of plastic consumes 2 tons of coal and 50 barrels of water. Organize these facts into matrices, and find the matrix that tells you how much coal ($c$) and water ($w$) is needed to make $r$ refrigerators and $a$ airplanes.

This is the same as Question 4 of Test A of 2002.

Let $s$ and $p$ be the quantities of steel and plastic, and let

$$\begin{pmatrix} s \\ p \end{pmatrix} = B \begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 0.1 & 5 \\ 0.2 & 2 \end{pmatrix} \begin{pmatrix} r \\ a \end{pmatrix}, \quad \begin{pmatrix} c \\ w \end{pmatrix} = A \begin{pmatrix} s \\ p \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 10 & 50 \end{pmatrix} \begin{pmatrix} s \\ p \end{pmatrix}.$$

Then $\begin{pmatrix} c \\ w \end{pmatrix} = AB \begin{pmatrix} r \\ a \end{pmatrix}$, where

$$AB = \begin{pmatrix} 3 & 2 \\ 10 & 50 \end{pmatrix} \begin{pmatrix} 0.1 & 5 \\ 0.2 & 2 \end{pmatrix} = \begin{pmatrix} 0.7 & 19 \\ 11 & 150 \end{pmatrix}.$$

(B) Calculate $\frac{d}{dx} \int_x^{x^2} \frac{\cos(tx)}{t} \, dt$.

By the multivariable chain rule, there will be three terms, one for each appearance of $x$. Two of them involve applications of the fundamental theorem of calculus, with a sign change when the lower limit is involved:

$$\cos(\frac{x^3}{x^2}) - \frac{\cos(x^2)}{x}.$$

The third term involves the derivative of the integrand, and it leaves an integral that can be evaluated:

$$\int_x^{x^2} (-\sin(tx)) \, dt = -\frac{1}{x} \int_x^{x^3} \sin z \, dz = \frac{\cos(x^3) - \cos(x^2)}{x}.$$

Adding, we get

$$\frac{3 \cos(x^3) - 2 \cos(x^2)}{x}.$$
7. (20 pts.) Find the inverse (if it exists) of the matrix \( M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 4 & 3 \end{pmatrix} \).

\[
\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{(1) \rightarrow (2)} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & -2 & 1 \end{pmatrix} \xrightarrow{(3) \rightarrow (3) - 2(1)} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & -4 & -2 & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & 0 & 0 & -\frac{4}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{4}{3} & -\frac{2}{3} & -\frac{1}{3} \end{pmatrix}
\]

Thus

\[
M^{-1} = \frac{1}{3} \begin{pmatrix} -4 & 1 & 1 \\ -1 & -2 & 1 \\ 4 & 2 & -1 \end{pmatrix}.
\]

This can easily be checked:

\[
\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 4 & 3 \end{pmatrix} \begin{pmatrix} -4 & 1 & 1 \\ -1 & -2 & 1 \\ 4 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.
\]