Math. 311 (Fulling) 24 October 2012

Test B – Solutions

Name: ____

Calculators may be used for simple arithmetic operations only!

1. *(22 pts.)* Let $\mathcal{V}$ be the span of the functions $\{e^{-x}, xe^{-x}, x^2e^{-x}\}$. Define $L(f) = f' + f$ (the prime means derivative). Note that L maps $\mathcal{V}$ into itself.

   (a) Find the matrix M representing L with respect to the basis $\{e^{-x}, xe^{-x}, x^2e^{-x}\}$ (used in both domain and codomain).

$$L(e^{-x}) = e^{-x}(-1+1) = 0\,,$$
$$L(xe^{-x}) = e^{-x}(-x+1+x) = e^{-x}\,,$$
$$L(x^2e^{-x}) = e^{-x}(-x^2+2x+x^2) = 2xe^{-x}\,.$$

So

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

   (b) Find the kernel of L.

In terms of coefficients, we must solve $M\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. That is, $b = 0$, $c = 0$, so that only a is nonzero. Thus in terms of functions, $\ker L$ consists of the multiples of the first basis element, e^{-x}.

Remark: We can also solve the differential equation $f' + f = 0$ directly, getting $f = Ce^{-x}$. These functions are in $\mathcal{V}$, so they are in the kernel (and no other functions can be).

   (c) Find the range of L.

It is clear from the matrix that $\operatorname{ran} L = \operatorname{span}\{e^{-x}, xe^{-x}\}$, or the functions $p(x)e^{-x}$ where p is a linear polynomial.

Remark: From the ODE point of view, if $L(p) = q$ and q contains a term proportional to x^2e^{-x}, then p must contain a term proportional to x^3e^{-x}, which is not in $\mathcal{V}$ and hence not allowed here.

2. *(10 pts.)* Find a basis for the span of the vectors $\left\{ \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} \right\}$.

Let's write the vectors as rows, writing the simplest one at the top to simplify the arithmetic:

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & -2 \\ 5 & 2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & -5 \\ 0 & -3 & -5 \end{pmatrix}.$$

Here we can stop, since the last row is redundant and the first two are obviously independent. A basis is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix} \right\}.$$

(There are others, of course.)

3. *(20 pts.)* The matrix of $L\colon \mathbf{R}^2 \to \mathbf{R}^2$ with respect to the natural basis is $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

   Find the matrix of L with respect to the basis $\left\{ \vec{b}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \vec{b}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$. (Change the basis in both domain and codomain.)

The matrix obtained by stacking the new basis elements together,

$$C = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

clearly maps the coordinates to the natural coordinates. Its inverse,

$$C^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

maps the coordinates the other way. So the desired matrix is

$$C^{-1}MC = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

4. *(28 pts.)* The matrix $M = \begin{pmatrix} 1 & 3 \\ 2 & -1 \\ 4 & 1 \end{pmatrix}$ represents a linear function $L\colon \mathbf{R}^2 \to \mathbf{R}^3$.

   (a) Is L surjective (onto)? If not, what is its range?

No. There are only 2 columns, so they can't possibly span $\mathbf{R}^3$. The range is the span of the columns,

$$\left\{ s \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

A slightly simpler basis for the range can be found by row reduction:

$$\left\{ x \begin{pmatrix} 1 \\ 0 \\ \frac{6}{7} \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ \frac{11}{7} \end{pmatrix} \right\}.$$

(b) Is L injective (one-to-one)? If not, what is its kernel?

Yes. $\dim \ker L = \dim \operatorname{dom} L - \dim \operatorname{ran} L = 2 - 2 = 0$. Alternatively, the matrix reduces to the echelon form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

so the homogeneous system has no nonzero solutions.

(c) What is the rank of L? What other ranks are possible for matrices of this size and shape?

The rank is 2, the dimension of the range. Other possible ranks are 1 (when the two columns are proportional) and 0 (when the whole matrix is zero).

(d) What matrix represents L if we switch to the basis

$$\left\{ \vec{b}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{b}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \vec{b}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

for $\mathbf{R}^3$?

In the notation used in the solution to Qu. 3,

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

In this problem we need $C^{-1}M$. A scandalously easy row reduction yields

$$C^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Thus

$$C^{-1}M = \begin{pmatrix} 2 & -1 \\ -4 & -1 \\ 1 & 3 \end{pmatrix}.$$

5. *(Essay — 20 pts.)* Do **ONE** of these [(A) or (B)]. (Half extra credit for doing the other one.)

(A) Define *subspace*. Then prove that for any linear function L, the kernel and range of L are subspaces (of what?).

(B) Explain how the ordinary differential equation $y''(t) + y(t) = \sin(2t)$ and its solution set exemplify the linear algebra concepts

linear, homogeneous, subspace, affine, range, kernel, superposition.