Math. 311 (Fulling) 26 November 2012

Test C – Solutions

Name: ______________________________

Calculators may be used for simple arithmetic operations only!

1. *(20 pts.)* Let $\mathbf{F}(\mathbf{r}) = 2xyz\,\hat\imath + (x^2z - z^2)\hat\jmath + (x^2y - 2yz)\hat k$.

(a) Find a potential $V(\mathbf{r})$ so that $\mathbf{F} = -\nabla V$.

Method 1: We have

$$-\frac{\partial V}{\partial x} = 2xyz\,, \quad -\frac{\partial V}{\partial y} = x^2z - z^2\,, \quad -\frac{\partial V}{\partial z} = x^2y - 2yz\,.$$

Therefore,

$$-V = x^2yz + C_1(y,z) = y(x^2z - z^2) + C_2(z,x) = x^2yz - yz^2 + C_3(x,y)\,.$$

Subtracting the term common to all three expressions, we find that

$$C_1(y,z) = -yz^2 + C_2(z,x) = -yz^2 + C_3(x,y)\,.$$

The only way these equations can be consistent is if $C_1(y,z) = -yz^2 + \text{constant}$. Thus (since we can set the constant to 0)

$$V(x,y,z) = -x^2yz + yz^2\,.$$

Method 2 (the hard way): Calculate

$$\nabla\times\mathbf{F} = \hat\imath(x^2 - z - x^2 + 2z) + \hat\jmath(2xy - 2xy) + \hat k(2xz - 2xz) = 0\,.$$

This assures us that $V(\mathbf{r})$ exists and equals $-\int_C \mathbf{F}\cdot d\mathbf{r}$ along any path from (say) the origin to $\mathbf{r}$. The easiest choices consist of line segments parallel to the axes; the second solution to part (b) is a simple special case.

(b) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ when C is the curve segment

$$x(t) = \sin(\pi t)\,, \quad y(t) = t^2\,, \quad z(t) = 1 + t^3 - t\,, \qquad 0 < t < 1\,.$$

First note that C begins at $\mathbf{r}_i = (0,0,1)$ and ends at $\mathbf{r}_f = (0,1,1)$.

Method 1: Use the result of part (a), first method:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = -V(\mathbf{r}_f) + V(\mathbf{r}_i) = -1 + 0 = -1\,.$$

Method 2: From part (a) we know that the integral is independent of path (either because V exists, or because $\nabla\times\mathbf{F} = 0$). So let's use the path

$$x = 0\,, \quad z = 1\,, \quad 0 \le y \le 1\,,$$

with y as the parameter. Only the dy term will be nonzero, since x and z are constant ($\frac{dx}{dy} = 0 = \frac{dz}{dy}$):

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 F_y\,dy = \int_0^1 (-1)\,dy = -1\,.$$

Method 3 (the hard way): $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left(F_x\frac{dx}{dt} + F_y\frac{dy}{dt} + F_z\frac{dz}{dy}\right)dt = \dots$ etc.

2. *(12 pts.)* Let $\mathcal{W}$ be the vector space of (real-valued, continuous) functions on the interval $0 \le t \le \pi$. Determine whether

$$\langle f, g\rangle = \int_0^\pi \frac{f(t)g(t)}{16 - t^2}\, dt$$

is an inner product on $\mathcal{W}$. Justify your answer in detail (from the definition of "inner product".

All 4 conditions in the definition are satisfied:

1. Symmetry: $\langle f, g\rangle = \langle g, f\rangle$ (obvious).
2. Bilinearity: $\langle \lambda f_1 + f_2, g\rangle = \lambda\langle f_1, g\rangle + \langle f_2, g\rangle$ (because integration is linear).
3. Positivity: The denominator is always positive because $t^2 \le \pi^2 \approx 10 < 16$. Therefore,

$$\langle f, f\rangle = \int_0^\pi \frac{f(t)^2}{16 - t^2}\, dt \ge 0\,.$$

4. Definiteness: Because f is continuous, $\langle f, f\rangle$ is not exactly 0 unless $f(t) = 0$ for all t.

3. *(18 pts.)* Find an orthonormal basis for the span of the vectors $\left\{ \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}3\\5\\7\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix} \right\}$.

We can start with the unit vector $\hat{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$. Then the perpendicular part of the second vector is

$$\vec{v}_{2\perp} = \begin{pmatrix}3\\5\\7\\0\end{pmatrix} - \frac{3+7}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}-2\\5\\2\\0\end{pmatrix}.$$

The associated unit vector is $\frac{1}{\sqrt{33}}\begin{pmatrix}-2\\5\\2\\0\end{pmatrix}$. Now the perpendicular part of the third vector is

$$\vec{v}_{3\perp} = \begin{pmatrix}0\\1\\0\\1\end{pmatrix} - 0\,\hat{u}_1 - \frac{5}{33}\begin{pmatrix}-2\\5\\2\\0\end{pmatrix} = \frac{1}{33}\begin{pmatrix}10\\8\\-10\\33\end{pmatrix},$$

and the corresponding unit vector is $\hat{u}_3 = \frac{1}{\sqrt{1353}}\begin{pmatrix}10\\8\\-10\\33\end{pmatrix}$.

Better method: Nothing in the problem (unlike the construction of orthogonal polynomials) requires us to go through the list of vectors in order. Since the third vector is already orthogonal to the first, we can choose our second unit vector to lie along the third of the original vectors: $\hat{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$. Now the perpendicular part of the remaining vector is

$$\begin{pmatrix} 3 \\ 5 \\ 7 \\ 0 \end{pmatrix} - \frac{10}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{5}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ \frac{5}{2} \\ 2 \\ -\frac{5}{2} \end{pmatrix}.$$

So

$$\hat{u}_3 = \frac{1}{\sqrt{82}} \begin{pmatrix} -4 \\ 5 \\ 4 \\ -5 \end{pmatrix}.$$

Checking that these answers are equivalent: We have constructed two different orthogonal bases for the subspace orthogonal to $\vec{u}_1$. When I put each of them into matrices and row-reduced, I got the same result:

$$\begin{pmatrix} -4 & 5 & 4 & -5 \\ 0 & 1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & \frac{5}{2} \\ 0 & 1 & 0 & 1 \end{pmatrix} \longleftarrow \begin{pmatrix} 10 & 8 & -10 & 33 \\ -2 & 5 & 2 & 0 \end{pmatrix}.$$

This verifies that the two sets span the same subspace. (The matrix in the middle displays yet a third basis for that subspace, which is "canonical" but not orthogonal.)

4. *(25 pts.)* Elliptic coordinates in $\mathbf{R}^2$ are defined by $\begin{cases} x = \cosh u \cos v, \\ y = \sinh u \sin v, \end{cases}$ with $0 \le u < \infty$ and $0 \le v < 2\pi$.

(a) Find the formulas for the tangent vectors to the coordinate curves.

$$\frac{d\vec{r}}{du} = \begin{pmatrix} \sinh u \cos v \\ \cosh u \sin v \end{pmatrix}, \qquad \frac{d\vec{r}}{dv} = \begin{pmatrix} -\cosh u \sin v \\ \sinh u \cos v \end{pmatrix}.$$

(b) Find the formulas (in terms of u and v) for the normal vectors to the coordinate curves.

For this part and the next we will need the determinant of the matrix formed out of the vectors in (a):

$$\frac{\partial(x,y)}{\partial(u,v)} = \sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v.$$

By Cramer's rule the inverse of that matrix is

$$\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^{-1} \begin{pmatrix} \sinh u \cos v & \cosh u \sin v \\ -\cosh u \sin v & \sinh u \cos v \end{pmatrix}.$$

The normal vectors are the rows of the inverse matrix:

$$\nabla u = \frac{(\sinh u \cos v, \cosh u \sin v)}{\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v}, \quad \nabla v = \frac{(-\cosh u \sin v, \sinh u \cos v)}{\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v}.$$

(c) Find the area of the region bounded by the curves $u = 1$, $u = 2$, $v = 0$, and $v = \frac{\pi}{2}$. (Set up the integral, don't evaluate it.)

Insert the Jacobian determinant:

$$\int_1^2 du \int_0^{\pi/2} dv\,(\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v)\,.$$

5. *(15 pts.)* Let $\mathbf{B}(\mathbf{r}) = xz\,\hat{\imath} + yz\,\hat{\jmath} + z^2\,\hat{k}$. Let S be the (open) hemisphere

$$r = 2\,, \quad 0 < \theta < \frac{\pi}{2}\,, \quad 0 < \phi < 2\pi\,.$$

Calculate $\iint_S \mathbf{B}\cdot d\mathbf{S}$.

Method 1: The normal vector to the sphere is $\hat{n} = \hat{r} = \frac{x}{r}\,\hat{\imath} + \frac{y}{r}\,\hat{\jmath} + \frac{z}{r}\,\hat{k}$. So

$$\mathbf{B}\cdot\hat{n} = \frac{z\mathbf{r}\cdot\mathbf{r}}{r} = zr\,,$$

which on S equals $2z = 4\cos\theta$. Therefore, the integral is

$$\int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi\,(4\cos\theta)r^2\sin\theta = (2\pi)16\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta = 16\pi\left(-\frac{1}{2}\right)\cos(2\theta)\Big|_0^{\pi/2} = 16\pi\,.$$

Method 2:

$$x = 2\sin\theta\cos\phi\,, \quad y = 2\sin\theta\sin\phi\,, \quad z = 2\cos\theta\,.$$

So

$$dx = 2\cos\theta\cos\phi\,d\theta - 2\sin\theta\sin\phi\,d\phi\,, \quad dy = 2\cos\theta\sin\phi\,d\theta + 2\sin\theta\cos\phi\,d\phi\,, \quad dz = -2\sin\theta\,d\theta\,.$$

So

$$dy\,dz = 4\sin^2\theta\cos\phi\,d\theta\,d\phi\,, \quad dz\,dz = 4\sin^2\theta\sin\phi\,d\theta\,d\phi\,,$$
$$dx\,dy = 4\cos\theta\sin\theta\cos^2\phi\,d\theta\,d\phi + 4\cos\theta\sin\theta\sin^2\phi\,d\theta\,d\phi\,.$$

So the integrand is

$$16\,d\theta\,d\phi\,[\sin^3\theta\cos\theta\cos^2\phi + \sin^3\theta\cos\theta\sin^2\phi + \cos^3\theta\sin\theta\cos^2\phi + \cos^3\theta\sin\theta\sin^2\phi]$$
$$= 16\,d\theta\,d\phi\,(\sin^3\theta + \cos^3\theta\sin\theta) = 16\,\sin\theta\cos\theta\,d\theta\,d\phi$$

just as before.

Method 3: We might try to apply Gauss's theorem, but the surface is not closed. However, we can make it into a closed surface by closing it off with a flat disc, D, at the bottom. The flux in through the disc, $\iint_D \mathbf{B}\cdot\hat{k}\,dS$, needs to be added to the volume integral. Miraculously, $\mathbf{B}\cdot\hat{k} = B_z = 0$ on the disc because $z = 0$ there. So we are left with just $\iiint \nabla\cdot\mathbf{B}\,dx\,dy\,dz$ over the solid hemisphere. Calculate $\nabla\cdot\mathbf{B} = z + z + 2z = 4z = 4r\cos\theta$. The integral becomes

$$\iiint \nabla\cdot\mathbf{B}\,dx\,dy\,dz = \int_0^2 dr\int_0^{\pi/2} d\theta\int_0^{2\pi} d\phi\,(r^2\sin\theta)(4r\cos\theta) = 2\pi\int_0^2 dr\int_0^{\pi/2} d\theta\,4r^3\sin\theta\cos\theta$$
$$= \frac{\pi}{2}(-1)\cos(2\theta)\Big|_0^{\pi/2}\int_0^2 4r^3\,dr = \pi\,r^4\Big|_0^2 = 16\pi.$$

6. *(10 pts.)* Evaluate the determinant $\begin{vmatrix} 1 & 0 & 1 & 0 \\ 5 & 9 & -1 & 2 \\ 0 & 3 & 2 & 0 \\ -1 & 0 & 1 & 2 \end{vmatrix}$.

$$\text{It} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 9 & -6 & 2 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{vmatrix} = (-6) \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 9 & -6 & 2 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= (-6) \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & -12 & 2 \\ 0 & 0 & 1 & 1 \end{vmatrix} = (-6)(-12-2) = 6 \cdot 14 = 84.$$

Of course, there are many other routes to the same answer.