Math. 311 (Fulling) 1 October 2014

Test A – Solutions

Calculators may be used for simple arithmetic operations only!

1. *(10 pts.)* Find a parametric representation of the plane through the point $(1, 3, 0)$ and parallel to the plane $x + y + z = 0$.

$$\vec{r} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + s\vec{u} + t\vec{v}$$

where $\vec{u}$ and $\vec{v}$ are in the parallel plane. (Note that that plane passes through the origin — otherwise we would need to subtract a fixed point from two other points to get vectors parallel to it.) These will do:

$$\vec{u} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

(Here I set $y = 0$, $z = 1$ and the reverse, and solved for x.)

Alternative solution (found on several student papers): Because the two planes have the same normal vector, the equation of the desired plane is

$$(x - 1) + (y - 3) + z = 0, \quad \text{or} \quad x + y + z = 4.$$

This is naturally solved as

$$\vec{r} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + s\vec{u} + t\vec{v}$$

with the same $\vec{u}$ and $\vec{v}$ as above.

2. *(22 pts.)* Solve these systems (introducing parameters when the answer is not unique).

(a) $$\begin{cases} x + 2y = 2, \\ 2x - 5y = 2. \end{cases}$$

Form and reduce the augmented matrix:

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & -5 & 2 \end{pmatrix} \xrightarrow{(2)\to(2)-2(1)} \begin{pmatrix} 1 & 2 & 2 \\ 0 & -9 & -2 \end{pmatrix} \xrightarrow[(1)\to(1)-2(2)]{(2\to-2(2)/9} \begin{pmatrix} 1 & 0 & \frac{14}{9} \\ 0 & 1 & \frac{2}{9} \end{pmatrix}.$$

Thus

$$x = \frac{14}{9}, \quad y = \frac{2}{9}.$$

The answer is easily checked by substitution.

(b) $$\begin{cases} x + 2y - 3z = 0, \\ 2x - 2y + 2z = 3. \end{cases}$$

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ 2 & -2 & 2 & 3 \end{pmatrix} \xrightarrow{(2)\to(2)-2(1)} \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & -6 & 8 & 3 \end{pmatrix} \xrightarrow{(2)\to(2)/6}$$

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & -\frac{4}{3} & -\frac{1}{2} \end{pmatrix} \xrightarrow{(1)\to(1)-2(2)} \begin{pmatrix} 1 & 0 & -\frac{1}{3} & 1 \\ 0 & 1 & -\frac{4}{3} & -\frac{1}{2} \end{pmatrix}.$$

Thus

$$x = 1 + \frac{z}{3}, \quad y = -\frac{1}{2} + \frac{4z}{3},$$

where z is arbitrary. Check by substitution.

3. *(12 pts.)* Construct the best affine approximation (also known as the first-order approximation) to

$$\vec{F} = \begin{pmatrix} x^2 - y \\ x^2 + y^2 \\ z^3 \end{pmatrix}$$

in the neighborhood of the point $\vec{r}_0 \equiv \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$. (Leave answer in matrix form.)

The Jacobian matrix is

$$\frac{d\vec{F}}{d\vec{r}} = \begin{pmatrix} 2x & -1 & 0 \\ 2x & 2y & 0 \\ 0 & 0 & 3z^2 \end{pmatrix} = \begin{pmatrix} 6 & -1 & 0 \\ 6 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \text{ at } \vec{r}_0 .$$

Therefore,

$$\vec{F}(\vec{r}) \approx \vec{F}(\vec{r}_0) + \frac{d\vec{F}}{d\vec{r}}(\vec{r} - \vec{r}_0) = \begin{pmatrix} 7 \\ 13 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 & -1 & 0 \\ 6 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x-3 \\ y-2 \\ z-1 \end{pmatrix}.$$

4. *(12 pts.)* The pressure at $\vec{r} = (x, y, z)$ is

$$P = 760 + x + yz \quad \text{torr}.$$

What is the rate of change of the reading on a pressure gauge that follows the path

$$\vec{r}(t) = (t+1, t^2, t)\,?$$

Use appropriate matrix or vector notation.

$$\frac{dP}{dt} = \frac{dP}{d\vec{r}} \cdot \frac{d\vec{r}}{dt} \equiv \nabla P\big(\vec{r}(t)\big) \cdot \vec{r}\,'(t).$$

$$\nabla P = (\,1 \quad z \quad y\,) = (\,1 \quad t \quad t^2\,); \qquad \frac{d\vec{r}}{dt} = \begin{pmatrix} 1 \\ 2t \\ 1 \end{pmatrix}.$$

So

$$\frac{dP}{dt} = (\,1 \quad t \quad t^2\,) \begin{pmatrix} 1 \\ 2t \\ 1 \end{pmatrix} = 1 + 2t^2 + t^2 = 1 + 3t^2.$$

5. *(10 pts.)* Define $[L(f)](t) = f'(t) + t^3 f(t-1)$. Prove or disprove that L is a linear operator (on functions f).

$$\begin{aligned} L(\lambda f_1 + f_2) &= (\lambda f_1 + f_2)'(t) + t^3(\lambda f_1 + f_2)(t-1) \\ &= \lambda f_1'(t) + f_2'(t) + \lambda t^3 f_1(t-1) + t^3 f_2(t-1) \\ &= \lambda f_1'(t) + \lambda t^3 f_1(t-1) + f_2'(t) + t^3 f_2(t-1) \\ &= \lambda L(f_1)(t) + L(f_2)(t)\,. \end{aligned}$$

6. *(14 pts.)* **Do ONE of these [(A) or (B)].** (Up to 7 points extra credit for doing the other one.)

(A) Producing a refrigerator requires 0.2 ton of steel and 0.2 ton of plastic. Producing an airplane requires 6 tons of steel and 2 tons of plastic. Producing a ton of steel consumes 4 tons of coal and 10 barrels of water. Producing a ton of plastic consumes 2 tons of coal and 30 barrels of water. Organize these facts into matrices, and find the matrix that tells you how much coal (c) and water (w) is needed to make r refrigerators and a airplanes.

Let s and p be the quantities of steel and plastic, and let

$$\begin{pmatrix} s \\ p \end{pmatrix} = B \begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 0.2 & 6 \\ 0.2 & 2 \end{pmatrix} \begin{pmatrix} r \\ a \end{pmatrix}, \qquad \begin{pmatrix} c \\ w \end{pmatrix} = A \begin{pmatrix} s \\ p \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 10 & 30 \end{pmatrix} \begin{pmatrix} s \\ p \end{pmatrix}.$$

Then $\begin{pmatrix} c \\ w \end{pmatrix} = AB \begin{pmatrix} r \\ a \end{pmatrix}$, where

$$AB = \begin{pmatrix} 4 & 2 \\ 10 & 30 \end{pmatrix} \begin{pmatrix} 0.2 & 6 \\ 0.2 & 2 \end{pmatrix} = \begin{pmatrix} 1.2 & 28 \\ 8 & 120 \end{pmatrix}.$$

(B) On the island of Lavonia the farmers keep $\frac{1}{2}$ of their produce and give $\frac{1}{4}$ to the blacksmiths and $\frac{1}{4}$ to the musicians. The blacksmiths give $\frac{2}{3}$ of their production to the farmers and keep the rest for themselves. The musicians can be heard by everyone, but because there are so few of them and so many farmers, the effective distribution of their benefits is $\frac{1}{2}$ to the farmers, $\frac{1}{3}$ to the blacksmiths, and $\frac{1}{6}$ to the musicians themselves. Write down the Leontief matrix, L, representing this economy, and write down (don't solve) the equation system determining the equilibrium production levels.

With the rows and columns labeled in the order shown, the matrix is

$$L = \begin{matrix} & F & B & M \\ & \downarrow & \downarrow & \downarrow \\ F \leftarrow & \frac{1}{2} & \frac{2}{3} & \frac{1}{2} \\ B \leftarrow & \frac{1}{4} & \frac{1}{3} & \frac{1}{3} \\ M \leftarrow & \frac{1}{4} & 0 & \frac{1}{6} \end{matrix}\,.$$

Equilibrium occurs if $L\vec{X} = \vec{X}$ — that is,

$$\begin{aligned} \tfrac{1}{2}F + \tfrac{2}{3}B + \tfrac{1}{2}M &= F, \\ \tfrac{1}{4}F + \tfrac{1}{3}B + \tfrac{1}{3}M &= B, \\ \tfrac{1}{4}F \qquad\quad + \tfrac{1}{6}M &= M. \end{aligned}$$

In other words, we need the solutions of $(L-I)\vec{X} = 0$, which we could find by row-reducing

$$\begin{pmatrix} -\frac{1}{2} & \frac{2}{3} & \frac{1}{2} \\ \frac{1}{4} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & 0 & -\frac{5}{6} \end{pmatrix}.$$

7. *(20 pts.)* Find the inverse (if it exists) of the matrix $M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ -2 & 4 & 3 \end{pmatrix}$.

Indicate what row operations you're performing.

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ -2 & 4 & 3 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{cycle rows}} \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ -2 & 4 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{(2)\to(2)+2(1)}$$

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 8 & 5 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{(2)\to(2)/8} \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{5}{8} & 0 & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{(1)\to(1)-2(2)}$$

$$\begin{pmatrix} 1 & 0 & -\frac{1}{4} & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 1 & \frac{5}{8} & 0 & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \xrightarrow[(2)\to(2)-5(3)/8]{(1)\to(1)+(3)/4} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{5}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Thus

$$M^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{5}{8} & \frac{1}{4} & \frac{1}{8} \\ 1 & 0 & 0 \end{pmatrix}.$$

Check by showing that $MM^{-1} = I$ or that $M^{-1}M = I$.