Math. 311 (Fulling) 29 October 2014

Test B – Solutions

Calculators may be used for simple arithmetic operations only!

1. *(20 pts.)* The linear function $F\colon \mathbf{R}^2 \to \mathbf{R}^2$ is represented by $A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ with respect to the natural basis.

(a) Find the matrix representing F if the basis in the domain is changed to

$$\mathcal{B} = \left\{ \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

(the basis for the codomain remaining unchanged).

The matrix $B \equiv \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ maps the $\mathcal{B}$-coordinates to the natural coordinates. Therefore, the desired matrix is

$$AB = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 4 & 2 \end{pmatrix}.$$

(b) Find the matrix representing F if the basis in the codomain is changed to $\mathcal{B}$ (the basis for the domain remaining unchanged).

Consider the matrix B from above. $B^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ maps natural coordinates to $\mathcal{B}$-coordinates. Therefore the desired matrix is

$$B^{-1}A = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 & 3 \\ -1 & 3 \end{pmatrix}.$$

(c) Find the matrix representing F if the basis $\mathcal{B}$ is used for both domain and codomain.

The correct matrix is $B^{-1}AB$. It can be calculated using the result of either (a) or (b) as an intermediate step. I'll do the latter:

$$(B^{-1}A)B = \frac{1}{2} \begin{pmatrix} 3 & 3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}.$$

2. *(24 pts.)* Let $\mathcal{V}$ be the vector space of quadratic polynomials (called $\mathcal{P}_2$ by Fulling but $\mathcal{P}_3$ by Leon). Let $L\colon \mathcal{V} \to \mathcal{V}$ be the differential operator $(Lp)(t) \equiv p''(t) + (t+1)p'(t) - p(t)$.

(a) Find the matrix that represents L with respect to the standard basis $\{t^2, t, 1\}$ for $\mathcal{V}$.

Calculate

$$\begin{aligned} L(t^2) &= 2 + (t+1)(2t) - t^2 = t^2 + 2t + 2\,, \\ L(t) &= 0 + (t+1) - t = 1\,, \\ L(1) &= 0 + 0 - 1 = -1\,. \end{aligned}$$

Put the coefficients of these polynomials into the columns of the matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 1 & -1 \end{pmatrix}.$$

(b) Find the kernel of L.

Row-reduce the matrix in (a) (see details in solution to (c)): $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. (We are interested in the solutions of the homogeneous equation, so we imagine a column of zeros on the right.) When a polynomial $p(t) = at^2 + bt + c$ is represented by the column $(a, b, c)^{\mathrm{T}}$, the column will be in the kernel of this matrix iff $b = c$ and $a = 0$. Thus the kernel of the polynomial problem is the multiples of $t + 1$.

(c) Find the range of L.

Method 1: Row-reduce the matrix, but this time augmented by an *arbitrary* column:

$$\begin{pmatrix} 1 & 0 & 0 & a \\ 2 & 0 & 0 & b \\ 2 & 1 & -1 & c \end{pmatrix} \xrightarrow{(i)\to(i)-2(1)} \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 0 & 0 & b-2a \\ 0 & 1 & -1 & c-2a \end{pmatrix}$$

$$\xrightarrow{(2)\leftrightarrow(3)} \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & -1 & c-2a \\ 0 & 0 & 0 & b-2a \end{pmatrix}.$$

The constraint that must be satisfied to get a solvable system is $b - 2a = 0$. So the range consists of polynomials of the form $at^2 + 2at + c$.

Method 2: Transpose the matrix of L and reduce:

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The rows give the coefficients of basis elements

$$\{t^2 + 2t, 1\}\,.$$

(d) Use "superposition principles" to find **all** solutions in $\mathcal{V}$ of the differential equation $p''(t) + (t+1)p'(t) - p(t) = 297.$

One obvious solution is $p(t) = -297$; $p(t) = +297t$ would also do. The general solution is that one plus the general solution of the homogeneous equation:

$$p(t) = C(t+1) - 297 = D(t+1) + 297t\,.$$

3. *(12 pts.)* Determine whether each set is linearly independent. If it is not, find an independent set with the same span.

(a) $$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$$

Yes. (If *two* vectors are dependent, then one is a multiple of the other.) *Alternative argument:* Reduce the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \end{pmatrix}$$

far enough to be sure that the rows are independent (span is 2-dimensional).

(b) $$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} \right\}$$

Yes. Let's transpose and row-reduce (so that the final rows answer the span question in case the answer is No):

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 1 & 0 & 5 \end{pmatrix} \xrightarrow{\text{reorder}} \begin{pmatrix} 1 & 0 & 5 \\ 1 & 2 & 3 \\ 2 & -1 & 4 \end{pmatrix} \xrightarrow[(3)\to(3)-2(1)]{(2)\to(2)-(1)} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 0 & -1 & -6 \end{pmatrix} \xrightarrow{(3)\to(3)+(2)/2} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 0 & 0 & -7 \end{pmatrix}.$$

It is now clear that the matrix is nonsingular, so the original vectors were independent.

Alternative argument: Calculate the determinant and observe that it is not zero.

4. *(Essay — 18 pts.)* Consider the two problems

$$\left.\begin{aligned} 3x - 2y &= 0\,, \\ 2x - 4y &= 2\,; \end{aligned}\right\} \qquad (1) \qquad\qquad \frac{d^2y}{dt^2} + 4y = e^t. \qquad (2)$$

Discuss the analogy between these problems and discuss what the principles of linear algebra tell us about their solutions. *Vocabulary hints:* linear, homogeneous, affine, subspace, kernel, range, superposition, …

5. *(20 pts.)* The matrix $M = \begin{pmatrix} 1 & -1 & 0 \\ 3 & 2 & 4 \end{pmatrix}$ represents a linear function $L\colon \mathbf{R}^3 \to \mathbf{R}^2$.

(a) Is L surjective (onto)? If not, what is its range?

Yes. (The range is all of $\mathbf{R}^2$.) Rationales: (1) The dimension of the range can't be 1 (or 0) because the columns are not all proportional. Or (2) Do (b) first, finding that the kernel has dimension 1 , so the range has dimension $3 - 1 = 2$.

(b) Is L injective (one-to-one)? If not, what is its kernel?

No, the linear function can't be injective when the matrix has more columns than rows. (In other words, when there are more unknowns than equations, the homogeneous system always has nontrivial solutions.) To find the kernel, row-reduce (omitting the augmenting column of zeros at the end):

$$\xrightarrow{(2)\to(3)-3(1)} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 5 & 4 \end{pmatrix} \xrightarrow[(1)\to(1)+(2)]{(2)\to(2)/5} \begin{pmatrix} 1 & 0 & \frac{4}{5} \\ 0 & 1 & \frac{4}{5} \end{pmatrix}.$$

So the solutions of the homogeneous equation have the form

$$z \text{ arbitrary}, \quad y = -\frac{4z}{5}, \quad x = -\frac{4z}{5}.$$

In other words, the kernel is

$$\operatorname{span}\left\{ \begin{pmatrix} -\frac{4}{5} \\ -\frac{4}{5} \\ 1 \end{pmatrix} \right\} \quad \text{or} \quad \operatorname{span}\left\{ \begin{pmatrix} 4 \\ 4 \\ -5 \end{pmatrix} \right\}.$$

(c) What is the rank of L ? What other ranks are possible for matrices of this size and shape?

2 (dimension of the range from (a), or $\dim \mathrm{dom} - \dim \ker = 3 - 1 = 2$ from (b)). The other possibilities are 0 and 1 .

6. *(6 pts.)* If $f(t) = c_1 e^{2t} + c_2 e^{-2t} = r_1 \cosh(2t) + r_2 \dfrac{\sinh(2t)}{2}$, what is the matrix M such that $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = M \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$?

We have

$$\cosh(2t) = \tfrac{1}{2} e^{2t} + \tfrac{1}{2} e^{-2t},$$
$$\tfrac{1}{2}\sinh(2t) = \tfrac{1}{4} e^{2t} - \tfrac{1}{4} e^{-2t}.$$

Therefore, $C = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}$ is the matrix expressing the hyperbolic basis vectors in terms of the exponential basis vectors. Therefore, its transpose is the matrix expressing the exponential coordinates in terms of the hyperbolic coordinates:

$$M = C^{\mathrm{T}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix}.$$

(Or, just substitute the formulas above into the formula for f , thereby reproving this theorem about the transpose for the particular case concerned.)