Vector Calculus and the Topology of Domains in 3-Space

Reference: J. Cantarella et al., *Amer. Math. Monthly* **109** (2002), 409–442.

The setting: Ω is a closed and bounded domain in $\mathbf{R}^3$ with smooth boundary $\partial\Omega$.

"Component" means a connected component (of Ω or $\partial\Omega$).

Question 4: Can you find a nonzero vector field on Ω that is divergence-free, curl-free, and normal to the boundary?

$\nabla\cdot\vec{V} = 0, \quad \nabla\times\vec{V} = \vec{0}, \quad \hat{n}\times\vec{V} = \vec{0}$ **(harmonic gradient)**

Note: If $\vec{V} = \nabla\phi$, then $\hat{n}\times\vec{V} = \vec{0}$ is equivalent to: ϕ is constant on each component of the boundary.

Answer 4: Such a vector field exists iff at least one component of Ω has more than one boundary component. (For example, Ω can be the region between two concentric spheres.) More precisely, the space of harmonic gradients is finite-dimensional with dimension equal to

$$(\#\text{ of components of }\partial\Omega) - (\#\text{ of components of }\Omega);$$

each such $\vec{V}$ is the gradient of a function ϕ, the constant values of ϕ on each boundary component are independent, and a different constant can be subtracted from ϕ on each component of Ω without changing $\vec{V}$.

Question 2: Given a vector field in Ω, how do you know whether it is the curl of another vector field?

$$\vec{V} = \nabla \times \vec{U}$$

Answer 2: Such a $\vec{U}$ exists iff $\nabla \cdot \vec{V} = 0$ and the flux of $\vec{V}$ through each boundary component is 0.

Remark (combining 4 and 2)**:** Any divergence-free vector field is the sum of a curl and a harmonic gradient. (Each harmonic gradient can be labeled by its fluxes through the boundary components,

$$F_{ij} = \int_{\partial\Omega_{ij}} \hat{n} \cdot \vec{V} \, dS,$$

where $\sum_j F_{ij} = 0$ by Gauss's theorem.)

Question 3: Can you find a nonzero vector field on Ω that is divergence-free, curl-free, and tangent to the boundary?

$$\nabla \cdot \vec{V} = 0, \quad \nabla \times \vec{V} = \vec{0}, \quad \hat{n} \cdot \vec{V} = 0 \qquad \textbf{(harmonic knot)}$$

Answer 3: Such a vector field exists iff at least one boundary component has nonzero genus. (For example, $\partial\Omega$ can be a torus.) More precisely, the space of harmonic knots is finite-dimensional with dimension equal to the genus (number of donut holes); each harmonic knot can be labeled by its line integral around each hole, or, equivalently, by its flux through each handle. (To be still more precise, we must learn what "first absolute homology group" and "second relative homology group" mean.)

Question 1: Given a vector field in Ω, how do you know whether it is the gradient of a function?

$$\vec{V} = \nabla \phi$$

Answer 1: Such a ϕ exists iff $\nabla \times \vec{V} = \vec{0}$ and the line integral of $\vec{V}$ around each hole is 0. (By Stokes's theorem, it doesn't matter which path around the hole you consider.)

Remark (combining 3 and 1)**:** Any curl-free vector field is the sum of a gradient and a harmonic knot.

VF(Ω) is the (infinite-dimensional) vector space of all smooth vector fields with domain Ω, equipped with the inner product $\langle \vec{V}, \vec{W} \rangle = \int_\Omega \vec{V}(\vec{r}) \cdot \vec{W}(\vec{r})\, d^3r$.

Hodge Decomposition Theorem: VF(Ω) is the direct sum of 5 mutually orthogonal subspaces:

$$\mathrm{VF} = \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG},$$

$$\begin{aligned}\mathrm{FK} &= \text{fluxless knots} \\ &= \{\nabla \cdot \vec{V} = 0,\ \hat{n} \cdot \vec{V} = 0,\ \text{handle fluxes} = 0\}\end{aligned}$$

$$\mathrm{HK} = \text{harmonic knots} = \{\nabla \cdot \vec{V} = 0,\ \nabla \times \vec{V} = \vec{0},\ \hat{n} \cdot \vec{V} = 0\}$$

$$\begin{aligned}\mathrm{CG} &= \text{curly gradients} \\ &= \{\vec{V} = \nabla\phi,\ \nabla \cdot \vec{V} = 0,\ \text{boundary fluxes} = 0\}\end{aligned}$$

$$\mathrm{HG} = \text{harmonic gradients} = \{\vec{V} = \nabla\phi,\ \nabla \cdot \vec{V} = 0,\ \hat{n} \times \vec{V} = 0\}$$

$$\mathrm{GG} = \text{grounded gradients} = \{\vec{V} = \nabla\phi,\ \phi = 0 \text{ on boundary}\}$$

Note: "Curly" means that $\vec{V}$ is the curl of something else, not that $\vec{V}$ itself has nonzero curl.

Moreover,

$$
\begin{aligned}
\ker \operatorname{curl} &= \phantom{\mathrm{FK} \oplus {}} \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG}, \\
\operatorname{ran} \operatorname{grad} &= \phantom{\mathrm{FK} \oplus \mathrm{HK} \oplus {}} \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG}, \\
\operatorname{ran} \operatorname{curl} &= \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \phantom{{} \oplus \mathrm{HG} \oplus \mathrm{GG}}, \\
\ker \operatorname{div} &= \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \phantom{{} \oplus \mathrm{GG}},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{FK} &= (\ker \operatorname{curl})^\perp, \\
\mathrm{HK} &= (\ker \operatorname{curl}) \cap (\operatorname{ran} \operatorname{grad})^\perp, \\
\mathrm{CG} &= (\operatorname{ran} \operatorname{grad}) \cap (\operatorname{ran} \operatorname{curl}), \\
\mathrm{HG} &= (\ker \operatorname{div}) \cap (\operatorname{ran} \operatorname{curl})^\perp, \\
\mathrm{GG} &= (\ker \operatorname{div})^\perp.
\end{aligned}
$$

Remark: "Knots" in general are $K = \{\nabla \cdot \vec{V} = 0,\ \hat{n} \cdot \vec{V} = 0\}$. They represent incompressible fluid flow through Ω with no vacuum or penetration at the boundary.

Let $\mathrm{G} = \{\vec{V} = \nabla\phi\},\ \mathrm{DFG} = \{\vec{V} = \nabla\phi,\ \nabla \cdot \vec{V} = 0\}$.

Step 1: $\mathrm{VF} = \mathrm{K} \oplus \mathrm{G}$ (orthogonal direct sum)

Step 2: $\mathrm{K} = \mathrm{FK} \oplus \mathrm{HK}$

Step 3: $\mathrm{G} = \mathrm{DFG} \oplus \mathrm{GG}$

Step 4: $\mathrm{DFG} = \mathrm{CG} \oplus \mathrm{HG}$

Steps 2 and 4 are hard, because they require construction of all the harmonic vector fields and relating them to the topology (as we outlined at the start).

Steps 1 and 3 are more elementary.