Math. 311 (Fulling) 24 March 2004

Test B – Solutions

Name: ____

Calculators may be used for simple arithmetic operations only!

1. *(16 pts.)* Let $\mathcal{M}$ be the vector space of 3×3 matrices, and let $\mathcal{S}$ be the subset of such matrices whose elements satisfy $A_{31} = A_{12} + A_{23}$. An example of a member of $\mathcal{S}$ is

$$\begin{pmatrix} 0 & 3 & 0 \\ -3 & -2 & 4 \\ 7 & 1 & 20 \end{pmatrix}.$$

(a) Prove that $\mathcal{S}$ is a *subspace* of $\mathcal{M}$.

$\mathcal{S}$ is the solution space (within $\mathcal{M}$) of the homogeneous linear equation $A_{31} = A_{12} + A_{23}$, so it is a subspace.

More explicitly, we can check that the defining condition is preserved under addition and scalar multiplication: Let A, B, and M be three matrices such that $M = rA + B$ for some number r. Then

$$M_{31} = rA_{31} + B_{31} = r(A_{12} + A_{23}) + B_{12} + B_{23} = (rA_{12} + B_{12}) + (rA_{23} + B_{23}) = M_{12} + M_{23}.$$

(b) The dimension of the vector space $\mathcal{M}$ is 9. Explain why.

There are 9 free parameters, the elements of the matrix. More explicitly, the most natural basis for $\mathcal{M}$ consists of the 9 matrices $\{E_{jk}\}_{j=1}^{3}\,{}_{k=1}^{3}$, where, for instance,

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(c) What is the dimension of the subspace $\mathcal{S}$?

The constraint means that there is one fewer parameter (e.g., A_{31} is determined once A_{12} and A_{23} are known), so the dimension is 8. More explicitly, a nice basis consists of the 2 matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and the 6 matrices E_{jk} corresponding to the matrix elements that are not involved in the constraint equation.

2. *(25 pts.)* The linear function $F\colon \mathbf{R}^2 \to \mathbf{R}^2$ is represented by $A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ with respect to the natural basis.

(a) Find the kernel of F. Is F one-to-one?

Reduce the augmented matrix for the equation system $F(\vec{r}) = \vec{0}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \to \cdots .$$

It is obvious now that the kernel contains only the zero vector: $\ker F = \{\vec{0}\}$. Therefore YES, F is one-to-one.

(b) Find the range of F. Is F onto?

Clearly the span of the columns (the range) is all of $\mathbf{R}^2$. Therefore YES, F is onto.

Alternatively, appeal to the theorem that for a square matrix, one-to-one and onto are equivalent (dimension of range = dimension of domain − dimension of kernel = $2 - 0 = 2$).

(c) Find the matrix representing F if the basis in the domain is changed to

$$\mathcal{B} = \left\{ \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

(the basis for the codomain remaining unchanged).

Matrix $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ maps $\mathcal{B}$-coordinates to natural coordinates, so the matrix we need is

$$AB = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 5 & 1 \end{pmatrix}.$$

(d) Find the matrix representing F if the basis $\mathcal{B}$ is used for both domain and codomain.

Note from (c) that $B^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. We need

$$M = B^{-1}AB = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}.$$

Check: $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, and

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

which confirms the columns in M.

3. *(12 pts.)* Determine whether each set is linearly independent. If it is not, find an independent set with the same span.

(a) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}$

Write the vectors as rows and reduce:

$$\begin{pmatrix} 1 & 2 & 4 \\ 3 & 3 & 6 \\ -1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ -1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix} \to \cdots$$

We see that there are only two independent rows. Therefore, the original set is NOT linearly independent, and one choice of basis is (from the rows of the completely reduced matrix)

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

(b) $\{\sinh(2t), \cosh(2t)\}$

This set is well known to be independent. (That's all I expected you to say.) To be more explicit, one could express it as

$$\sinh(2t) = \frac{1}{2}e^{2t} - \frac{1}{2}e^{-2t}, \quad \cosh(2t) = \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t}$$

in terms of a set $\{e^{2t}, e^{-2t}\}$ whose independence is even more well known, and observe that the rows of

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

are independent. Or, go even further back to first principles and choose two values of t, say 0 and 1, and evaluate the functions there:

$$(0, 1), \quad (\sinh 2, \cosh 2);$$

since these numerical vectors are independent, it is impossible for the functions to be dependent (i.e., $c_1 \sinh(2t) + c_2 \cosh(2t) = 0$ can't be true for all t, because it is false for $t = 0$ and 1). (*Note:* This kind of argument is not valid in the converse direction! Consider the set $\{\sin t, \cos t\}$ with $t = 0$ and π.)

4. *(Essay — 20 pts.)* Consider the two problems

$$\left.\begin{aligned} x - 2y &= 1\,, \\ 2x - 4y &= 2\,; \end{aligned}\right\} \qquad (1) \qquad\qquad \frac{d^2y}{dt^2} = e^t. \qquad (2)$$

Discuss the analogy between these problems and discuss what the principles of linear algebra tell us about their solutions. *Vocabulary hints:* linear, homogeneous, affine, subspace, kernel, range, superposition, …

I tried to give points for each valid observation made, somewhat like this:

1	Both problems are linear
1	and nonhomogeneous,
2	so their solution spaces are affine subspaces,
1	which by superposition principles
2	can be constructed as $y = y_{\rm p} + y_{\rm h}$,
1	(that is, (particular solution) + (kernel)).
2	In both problems the kernel is nontrivial
1	(i.e., their linear operators are not injective),
1	and hence the solutions of the nonhomogeneous problems are not unique.
1	In (1) the operator is not surjective either,
1	but $\binom{1}{2}$ is in the range regardless,
1	and therefore solutions to (1) exist.
1	The general solution of (1) is $\binom{x}{y} = \binom{2y+1}{y}$,
2	which fits into the general framework with $y_{\rm p} = \binom{1}{0}$, kernel = $\left\{ y\binom{2}{1} \right\}$.
1	In (2) the operator *is* surjective ($y'' = f$ can be solved for any reasonable function f),
1	so solutions to (2) exist.
2	(To be more precise one should choose domain = $\mathcal{C}^2(a,b)$, codomain = $\mathcal{C}(a,b)$, for some interval (a,b) .)
1	A particular solution of (2) is $y_{\rm p} = e^t$
2	and the general element of the kernel is $y_{\rm h} = c_1 t + c_2$.

This gives as many as 25 points. Occasionally points were taken off for false remarks and confused use of terminology. The maximum actual score was 16, so I added 5 points to the test scores at the end.

5. *(27 pts.)* Let $L\colon \mathcal{P}_2 \to \mathcal{P}_1$ be the differential operator $[Lp](t) \equiv tp''(t) - p'(t)$.

(a) Find the matrix that represents L with respect to the standard bases $\{t^2, t, 1\}$ for $\mathcal{P}_2$ and $\{t, 1\}$ for $\mathcal{P}_1$.

Calculate

$$L(t^2) = 2t - 2t = 0, \quad L(t) = -1, \quad L(1) = 0.$$

By the kth column rule, the matrix is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

(b) Find the kernel of L . Is L injective?

NO. If we write $p(t) = at^2 + bt + c$, then it is clear from the matrix in (a) that the kernel is the cases with $b = 0$ but a and c arbitrary: $\ker L = \{at^2 + c\}$.

Alternatively, we can solve the differential equation for $q \equiv p'$:

$$\frac{dq}{q} = \frac{dt}{t} \;\Rightarrow\; \ln q = \ln t + C_1 \;\Rightarrow\; q = C_2 e^{\ln t} = C_2 t \;\Rightarrow\; p = Ct^2 + D$$

(where I kept redefining the arbitrary constant to keep each expression maximally simple). Thus all solutions turn out to be quadratic polynomials, precisely the same ones we found by the matrix method.

(c) Find the range of L. Is L surjective?

NO. It is immediately clear from (b) that the dimension of the range is $3-2=1$. And it is clear from the matrix that the range consists of the constant functions. (This statement applies when the domain is chosen to be $\mathcal{P}_2$. If the domain is the whole function space $\mathcal{C}^2$, for example, of course there are many other functions in the range.)

(d) Find the matrix that represents L with respect to the standard basis $\{t^2, t, 1\}$ for $\mathcal{P}_2$ and the basis $\{t, t+1\}$ for $\mathcal{P}_1$.

We are changing basis in the codomain, so we need to multiply by a matrix M on the left. Observe that

$$C \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

maps the old basis vectors to the new basis vectors. Therefore, $M \equiv (C^{\mathrm{t}})^{-1}$ maps the old coordinates to the new coordinates. We easily find that

$$M = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Therefore, the desired matrix is

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

We can easily check this by the kth column rule:

$$L(t^2) = 0, \quad L(t) - 1 = -(t+1) + t, \qquad L(1) = 0.$$