14.4

(a) Solve the wave equation in $\mathbf{R}^2$ by Fourier transforms, and rearrange the result into the form

$$u(x,t) = \int_{-\infty}^{\infty} dy\, [W(x-y,t)f(y) + V(x-y,t)g(y)],$$

where f and g are the initial data (in the notation of the printed notes), and W and V are certain integral expressions.

(b) What identities for the Fourier transforms of delta functions and step functions do you need to assume in order to get your answer to agree with the one on p. 134 of the notes? Try to boil it down to simple formulas for functions (or distributions) of *one* variable, not three.

(a) The wave equation is $u_{tt} = u_{xx}$ and the initial data are $u(0,x) = f(x)$, $u_t(0,x) = g(x)$. Take Fourier transforms with respect to x:

$$\hat{u}_{tt}(t,\omega) = -\omega^2 \hat{u}(t,\omega), \qquad \hat{u}(0,\omega) = \hat{f}(\omega), \quad \hat{u}_t(0,\omega) = \hat{g}(\omega).$$

Therefore,

$$\hat{u}(t,\omega) = A(\omega)\cos(\omega t) + B(\omega)\sin(\omega t),$$

and

$$A(\omega) = \hat{f}(\omega), \quad \omega B(\omega) = \hat{g}(\omega),$$

so

$$B(\omega) = \frac{g(\omega)}{\omega}.$$

Now plug in the definitions

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega y} f(y)\, dy, \quad \hat{g}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega y} g(y)\, dy$$

and

$$u(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\hat{u}(t,\omega)\, d\omega.$$

(No matter where you put the factors of $\sqrt{2\pi}$, there will be two of them in the final denominator.) The result is

$$u(t,x) = \int_{-\infty}^{\infty} dy\, \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(x-y)}\left[\cos(\omega t)f(y) + \frac{\sin(\omega t)}{\omega}\, g(y)\right].$$

So we can identify

$$W(x-y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(x-y)}\cos(\omega t),$$

$$V(x-y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(x-y)}\frac{\sin(\omega t)}{\omega}.$$

(b) Oops! The function called W in the notes is the one called V in the problem. Sorry! We know (from pp. 114–115 of the notes) that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega z} = \delta(z).$$

Thus we can evaluate W above as

$$W(x-y,t) = \tfrac{1}{2}[\delta(x-y+t) + \delta(x-y-t)],$$

or just

$$W(z,t) = \tfrac{1}{2}[\delta(z+t) + \delta(z-t)].$$

If you differentiate the formula for V with respect to t "formally" (that is, not worrying about whether the calculus operations make sense for these poorly convergent improper integrals), you see that the result is (the integral formula for) W:

$$W(z,t) = \frac{\partial V}{\partial t}(z,t).$$

Therefore, V ought to be some antiderivative of W. As explained in the notes, the antiderivate of a delta function is a step function, so we should have (for $t > 0$)

$$V(z,t) = C + \begin{cases} 0 & \text{for } z < -t, \\ \frac{1}{2} & \text{for } -t < z < t, \\ 1 & \text{for } z > t. \end{cases}$$

Furthermore, the integral formula for V shows that $V(z,t)$ is an odd function of z, so the constant of integration is $C = -\frac{1}{2}$. This gives the formulas in the notes (if you rename V as W, and do the right bookkeeping to get inequalities also valid for the case $t < 0$).