Math. 401 (Fulling) 31 October 2003

Test B – Solutions

1. *(40 pts.)* Solve by singular perturbation theory (for $0 < \epsilon \ll 1$)

$$\epsilon \frac{d^2y}{dx^2} - \frac{dy}{dx} + xy = 0 \qquad (0 < x < 1),$$

$$y(0) = 1, \qquad y(1) = 0.$$

(Find outer, inner, and composite approximations to lowest order.)

Because the first-order and second-order terms have opposite signs, we expect the boundary layer to be at the right end.

Outer solution: To zeroth order we must have $-y' + xy = 0$, or

$$\frac{dy}{y} = x\,dx \quad \Rightarrow \quad \ln y = \frac{1}{2}x^2 + c \quad \Rightarrow \quad y = Ce^{x^2/2}.$$

Impose the left-end boundary condition: $1 = y(0) = C$. Thus

$$y_{\mathrm{o}} = e^{x^2/2}.$$

Inner solution: Let $z = \dfrac{x-1}{\epsilon}$, so that $x = 1 + \epsilon z$ and z is negative inside the interval. Then $\frac{d}{dx} = \frac{1}{\epsilon}\frac{d}{dz}$, so the ODE transforms (after multiplication by ϵ) to

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + (\epsilon + \epsilon^2 z)y = 0.$$

So to zeroth order we must have $(y')' - y' = 0$, so $y' = Ae^z$, hence

$$y = Ae^z + B = Ae^{(x-1)/\epsilon} + B.$$

Impose the right-end boundary condition: $0 = y(1) = A + B$, so $B = -A$. Thus

$$y_{\mathrm{i}} = A\bigl[e^{(x-1)/\epsilon} - 1\bigr].$$

Composite solution: Let $\eta = \sqrt{\epsilon}\,z = \dfrac{x-1}{\sqrt{\epsilon}}$. Study the limit $\epsilon \to 0^+$ with η fixed (and negative).

$$y_{\mathrm{o}} = e^{(1+\sqrt{\epsilon}\,\eta)^2/2} \to e^{1/2}.$$

$$y_{\mathrm{i}} = A\bigl[e^{\eta/\sqrt{\epsilon}} - 1\bigr] \to -A.$$

Therefore, $A = -\sqrt{\epsilon}$, Construct the uniform, composite solution by adding the two nonuniform solutions and subtracting their common limit, $\sqrt{\epsilon}$:

$$\begin{aligned} y &\sim y_{\mathrm{o}} + y_{\mathrm{i}} - \sqrt{e} \\ &= e^{x^2/2} + \sqrt{e}\bigl[1 - e^{\frac{(x-1)}{\epsilon}}\bigr] - \sqrt{e} \\ &= e^{x^2/2} - \sqrt{e}\,e^{\frac{(x-1)}{\epsilon}}. \end{aligned}$$

Check:

$$y(0) = 1 - \sqrt{e}\, e^{-1/\epsilon} \sim 1, \qquad y(1) = e^{1/2} - \sqrt{e} = 0.$$

$$y' = xe^{x^2/2} - \frac{\sqrt{e}}{\epsilon}\, e^{(x-1)/\epsilon}, \qquad y'' = (1+x^2)e^{x^2/2} - \frac{\sqrt{e}}{\epsilon^2}\, e^{(x-1)/\epsilon},$$

so

$$\begin{aligned} \epsilon y'' - y' + xy &= \epsilon(1+x^2)e^{x^2/2} - \frac{\sqrt{e}}{\epsilon}\, e^{(x-1)/\epsilon} - xe^{x^2/2} + \frac{\sqrt{e}}{\epsilon}\, e^{(x-1)/\epsilon} + xe^{x^2/2} - x\sqrt{e}\, e^{(x-1)/\epsilon} \\ &= \epsilon(1+x^2)e^{x^2/2} - x\sqrt{e}\, e^{(x-1)/\epsilon} \end{aligned}$$

after cancellations. We observe that the leading terms of both the exponential and the nonexponential type have cancelled, which is exactly what should be expected from the zeroth-order approximation.

2. *(40 pts.)*

(a) Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < 1, \quad 0 < t < \infty),$$

$$u(t,0) = 0, \qquad u(t,1) = 2, \qquad u(0,x) = f(x).$$

(It is not necessary to rederive "well known" facts about Fourier series; just use them, making sure that your reasoning is clear.)

Since there is a nonhomogeneous boundary condition, we need to write

$$u = v + w, \tag{1}$$

where v is a steady-state solution and w satisfies homogeneous boundary conditions.

Steady-state solution: Let $v(t,x) = V(x)$ where

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial V}{\partial t} = 0, \qquad V(0) = 0, \quad V(1) = 2.$$

We have $V(x) = Ax + B$ and $0 = B$, $2 = A + B$, hence

$$v(t,x) = 2x. \tag{2}$$

Separation of variables: The rest of the solution must satisfy

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2}\,, \qquad w(t,0) = 0, \quad w(t,1) = 0, \qquad w(0,x) = g(x),$$

where

$$g(x) \equiv f(x) - 2x. \tag{3}$$

Look first for separated solutions, $w_{\text{sep}} = X(x)T(t)$. Then $XT' = X''T$, or

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda.$$

Thus we have the eigenvalue problem

$$X'' + \lambda X = 0, \qquad X(0) = 0, \quad X(1) = 0,$$

whose solutions are well known to be

$$X_n(x) = \sin(n\pi x), \qquad \lambda_n = (n\pi)^2, \qquad n = 1, 2, \ldots,$$

along with the equation $T' = -\lambda T$, whence

$$T_n(t) = e^{-\lambda_n t}.$$

Superpose these solutions:

$$w(t,x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-n^2\pi^2 t}. \tag{4}$$

This must satisfy the initial condition

$$g(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x),$$

from which we get

$$b_n = 2\int_0^1 \sin(n\pi x) g(x)\, dx. \tag{5}$$

Collecting all the numbered equations, we get the solution $u(t,x)$.

(b) What conditions would you impose on f to guarantee that the Fourier series converges uniformly, even when $t = 0$?

We want g to be continuous and piecewise smooth and to vanish at the endpoints. These translate into the same conditions on f, except that $f(1)$ must equal 2, not 0.

3. *(20 pts.)* Consider the eigenvalue problem (on $0 < x < \pi$)

$$-X''(x) + KX(x) = \lambda X(x), \qquad X(0) = 0 = X(\pi).$$

(a) Find the eigenvalues and eigenfunctions, supposing that K is a (real) *constant*.

This is just our familiar Fourier sine problem in disguise. We have $X(x) = \sin(nx)$ as usual, with the usual restrictions on n, but now $n^2 = \lambda - K$. Thus

$$\lambda_n = n^2 + K, \quad n = 1, 2, \ldots.$$

(b) Now suppose that K is a (real-valued, differentiable) *function* of x. Use the WKB approximation to argue that the eigenvalues are approximately the numbers λ_n that satisfy

$$\int_0^\pi \sqrt{\lambda_n - K(x)}\,dx = n\pi \qquad (n = 0, 1, \ldots).$$

The basic WKB approximation for this type of equation is

$$y \sim \frac{1}{\sqrt[4]{\lambda - K(x)}} \exp\left[\pm i \int \sqrt{\lambda - K(x)}\,dx\right].$$

From the boundary condition $X(0) = 0$ it is clear that the relevant solution is (arbitrary constant times)

$$\frac{1}{\sqrt[4]{\lambda - K(x)}} \sin\left[\int_0^x \sqrt{\lambda - K(\tilde{x})}\,d\tilde{x}\right],$$

and then the other condition, $X(\pi) = 0$, yields

$$\sin\left[\int_0^\pi \sqrt{\lambda - K(\tilde{x})}\,d\tilde{x}\right] = 0,$$

or

$$\int_0^\pi \sqrt{\lambda - K(x)}\,dx = n\pi.$$

Once $\lambda - K(x)$ is positive for all x, the left-hand side will be an increasing function of λ, so there should be exactly one solution, $\lambda = \lambda_n$, for each n, at least if n is sufficiently large (see (d)).

(c) Check the consistency of the results of (a) and (b).

If K is constant, we can evaluate the integral to get

$$n\pi = \pi\sqrt{\lambda_n - K},$$

which is equivalent to the result of (a). Also, the WKB formula for X is exact in this case.

(d) Would you expect (b) to give good approximations to the eigenvalues for all n? Just small n? Just large n? Or what?

Given that K is differentiable, the approximation should certainly be valid for all large n. At small n things *might* go wrong. First, $\lambda - K(x)$ needs to be large compared to its derivatives in order for the WKB approximation to $X(x)$ to be accurate. Moreover, for the really small eigenvalues we might get into the regime where $\lambda - K(x)$ is not even positive, so the WKB formula for X doesn't apply at all, and also the argument that there is a unique solution for λ_n breaks down. (Later, however, we will see from the Sturm–Liouville theory (counting the nodes) that the eigenvalue approximated by our λ_n (for any large n) really is the nth eigenvalue — not number $n + 2$ or something like that.)