MATH 401: Section 500 & 501 Spring 2001
Final Exam – Solutions

1.[20pts.] Below are listed four problems that can be solved using separation of variables and/or some combination of Fourier series and transforms.

(a) Laplace's equation in a rectangle $(0 < x < a,\ 0 < y < b)$ **C**
(b) Heat equation on a finite interval $(0 < x < L)$ **A**
(c) Laplace's equation in a semi-infinite vertical strip $(0 < x < l,\ 0 < y < \infty)$ **B**
(d) Heat equation in an infinite interval $(-\infty < x < \infty)$ **E**

Assume each problem has the maximum number of inhomogeneous boundary conditions. For each problem, pick the form of the expected solution from the following list.

(A) the sum of a steady-state solution and a Fourier series
(B) the sum of both a Fourier series and two Fourier sine or cosine transforms
(C) the sum of four Fourier series
(D) Fourier transforms with respect to two different variables
(E) a Fourier transform

2.[20pts.] Classify each of the following equations as linear homogeneous, linear inhomogeneous, or nonlinear. Also, classify each as elliptic, parabolic, or hyperbolic.

(A)$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} - xyu = -(x+y)$

Linear nonhomogeneous, elliptic.

(B)$\dfrac{\partial w}{\partial t} + \dfrac{\partial^2 w}{\partial x^2} = t^2 w$

Linear homogeneous, parabolic.

3.[40pts.] For the initial-value-problem

$$\frac{d^2y}{dt^2} + 4y + \epsilon y^3 = 0, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = 1,$$

find the first term of the regular expansion for the solution. Also, write down the equation you would use to find the second term in the expansion. DO NOT solve this equation, but use it to explain why the regular expansion will fail to be uniform. You may find helpful the trig identity

$$\sin^3 x = -\frac{1}{4}\sin 3x + \frac{3}{4}\sin x.$$

Finally, describe briefly what you would do to remedy the situation.

If $y \sim y_0 + \epsilon y_1$ then $y^3 \sim y_0^3 + 3\epsilon y_1 y_0^2$. The initial conditions translate to

$$y_0(0) = 0, \quad y_0'(0) = 1; \qquad y_1(0) = 0, \quad y_1'(0) = 0.$$

The equation becomes

$$0 \sim y_0'' + \epsilon y_1'' + 4y_0 + 4\epsilon y_1 + \epsilon y_0^3.$$

The ϵ^0 problem is

$$y_0'' + 4y_0 = 0 \quad \text{with initial conditions} \Longrightarrow y_0 = \tfrac{1}{2}\sin 2t.$$

The ϵ^1 problem is

$$y_1'' + 4y_1 = -y_0^3 = -\left(\tfrac{1}{2}\sin 2t\right)^3 = -\tfrac{1}{8}\sin^3 2t,$$

or

$$y_1'' + 4y_1 = \frac{1}{32}\sin 6t - \frac{3}{32}\sin 2t.$$

Since the $\sin 2t$ term is resonant, the solution for y_1 will contain secular terms like $t\sin 2t$; the approximation by regular perturbation theory is nonuniform. To improve the approximation by the distorted-time (Poincaré) method, try a new time scale $\tau \sim (1+\omega_1\epsilon)t$ and choose ω_1 to cancel the secular terms in the ODE for y_1. The improved y_0 is now a one-term approximation that already incorporates the effect of the perturbation to first order, in a more nearly uniform manner. The two-time method could also be used, but it is "overkill" for this problem.

4.[40pts.] Obtain a one-term uniformly valid composite expansion for the solution of

$$\epsilon\frac{d^2 f}{dx^2} - \frac{df}{dx} + xe^{-f} = 0, \quad 0 < x < 2,$$

$$0 < \epsilon \ll 1, \quad f(0) = 1, \quad f(2) = 3.$$

First find the lowest-order outer solution by just setting ϵ to 0 in the equation:

$$-\frac{df_0}{dx} + xe^{-f_0} = 0.$$

This equation is separable (in the sense of ODEs):

$$\int e^{+f_0}\,df_0 = \int x\,dx \Longrightarrow e^{f_0} = \frac{x^2}{2} + C \Longrightarrow f_0 = \ln\left(\frac{x^2}{2} + C\right).$$

The negative sign on the first derivative suggests a boundary layer on the right, so we enforce the left-end boundary condition $1 = f_0(0) = \ln C$, or $C = e$.

To construct the inner solution, let $s = (2-x)/\epsilon$, so that s vanishes at the right endpoint and is positive inside the interval. Then

$$\frac{d}{dx} = -\frac{1}{\epsilon}\frac{d}{ds}, \qquad x = 2 - \epsilon s,$$

so the ODE (multiplied by ϵ) converts to

$$\frac{d^2 f_i}{ds^2} + \frac{df_i}{ds} + O(\epsilon) = 0.$$

Thus

$$\frac{df_i}{ds} = Ae^{-s}, \qquad f_i = -Ae^{-s} + B.$$

The boundary condition (inherited from $x = 2$) is $3 = f_i(0) = -A + B$. So

$$f_i = (3 + A) - Ae^{(x-2)/\epsilon}.$$

Now construct the uniform approximation: Let

$$\eta = \sqrt{\epsilon} s = \frac{2-x}{\sqrt{\epsilon}} > 0.$$

$$f_i = (3+A) - Ae^{-\eta/\sqrt{\epsilon}} \to 3 + A;$$

$$f_0 = \ln\left(e + \frac{(2-\sqrt{\epsilon}\eta)^2}{2}\right) \to \ln(e+2).$$

Equate the limits: $A = \ln(e+2) - 3$. Add the two solutions and subtract the common limit ($\ln(e+2)$, which cancels a term in f_i):

$$f(x) \approx [3 - \ln(e+2)]e^{(x-2)/\epsilon} + \ln\left(e + \frac{x^2}{2}\right).$$

5.[40pts.] & 6.[40pts.] Do **ANY TWO** of these four problems. Because of the grading deadline, no more than two will be graded, so do ONLY TWO!
(A) Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \infty, \quad 0 < y < \infty,$$

$$u(x,0) = g(x), \qquad u(0,y) = 0, \qquad |u(x,y)| \leq \text{constant}.$$

Then show that the solution can be rewritten in the form

$$u(x,y) = \int_0^\infty G(x,y,z) g(z)\, dz.$$

For the second part of this problem, you may find one or the other of these formulas useful:

$$\int_{-\infty}^{\infty} e^{i\omega x}\, e^{-\omega^2 t}\, d\omega = \sqrt{\frac{\pi}{t}}\, e^{-x^2/4t}$$

$$\int_{-\infty}^{\infty} e^{i\omega(x-z)}\, e^{-|\omega| y}\, d\omega = \frac{2y}{(x-z)^2 + y^2}$$

Separate variables: $u_{\text{sep}}(x,y) = X(x)Y(y) \Longrightarrow X''Y + XY'' = 0 \Longrightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\omega^2$. We take the separation constant negative (yielding oscillatory solutions for X) because the boundary conditions suggest that a sine transform with respect to x will be needed: $X_\omega(x) = \sin(\omega x)$, $0 < \omega < \infty$. Since the solution must vanish (or at least remain "well-behaved") at infinity, the solution for Y must be a decaying exponential: $Y_\omega(y) = e^{-\omega y}$. Putting all this together, we get

$$u(x,y) = \int_0^\infty B(\omega) \sin(\omega x) e^{-\omega y}\, d\omega.$$

What remains is to satisfy the nonhomogeneous boundary condition:

$$g(x) = \int_0^\infty B(\omega) \sin(\omega x)\, d\omega,$$

hence

$$B(\omega) = \frac{2}{\pi}\int_0^\infty g(x)\sin(\omega x)\,dx.$$

For the second part, insert the formula for B, with x renamed z, into the formula for u, and exchange the order of integrations:

$$u(x,y) = \int_0^\infty dz \int_0^\infty d\omega\,\frac{2}{\pi}\,\sin(\omega x)\sin(\omega z)e^{-\omega y}.$$

This has the desired Green function form with

$$\begin{aligned} G(x,y,z) &= \int_0^\infty d\omega\,\frac{2}{\pi}\,\sin(\omega x)\sin(\omega z)e^{-\omega y} \\ &= \frac{-1}{2\pi}\int_0^\infty d\omega\,[e^{i\omega(x+z)} - e^{i\omega(x-z)} - e^{i\omega(z-x)} + e^{-i\omega(x+z)}]e^{-\omega y} \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty d\omega[e^{i\omega(x-z)} - e^{i\omega(x+z)}]e^{-|\omega|y} \\ &= \frac{y/\pi}{(x-z)^2+y^2} - \frac{y/\pi}{(x+z)^2+y^2}\,. \end{aligned}$$

(This is the same as the Green function for the entire upper half plane together with an "image charge" term to force the solution to zero at the vertical boundary of the quadrant.)

(B) Solve

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad 0 < t < \infty, \\ &\frac{\partial u}{\partial x}(t,0) = N, \qquad \frac{\partial u}{\partial x}(t,\pi) + \beta u(t,\pi) = C, \\ &u(0,x) = f(x), \end{aligned}$$

where N, C, and β are prescribed constants, β is positive, and f is a prescribed function. Cite any appropriate theorems needed to justify your answer.

Because of the nonhomogeneous, but time-independent, terms N and C, we need to peel off a steady-state solution. Let $u = v + w$, where $v = v(x)$ satisfies

$$v'' = 0, \qquad v'(0) = N, \quad v'(\pi) + \beta v(\pi) = C.$$

Then $v(x) = Ax + B$ where $N = A$ and $C = A + \beta(A\pi + B)$. Thus

$$v(x) = Nx + \frac{C-N}{\beta} - \pi N.$$

Write $h(x)$ for $f(x) - v(x)$. The other part of the solution, w, satisfies the homogeneous boundary-value problem

$$w_t = w_{xx}\,, \qquad w_x(t,0) = 0 = w_x(t,\pi) + \beta w(t,\pi), \qquad w(0,x) = h(x).$$

Assuming a separated solution $w = X(x)T(t)$ with an oscillatory eigenfunction (positive eigenvalue) leads to

$$\frac{T'}{T} = \frac{X''}{X} = -\omega^2.$$

The eigenvalue problem is

$$X'' + \omega^2 X = 0, \qquad X'(0) = 0, \quad X'(\pi) + \beta X(\pi) = 0.$$

The solutions are $X(x) = \cos(\omega x)$ with $-\omega \sin(\omega\pi) + \beta\cos(\omega\pi) = 0$. The allowed eigenvalues satisfy

$$\frac{\omega}{\beta} = \cot(\omega\pi).$$

The two sides of this equation can be easily sketched, showing that there are infinitely many positive roots ω_1, ω_2, …, each ω_j being greater than $(j-1)\pi$ and increasingly close to the latter as j increases. Negative ω's give nothing new. The general theory of Sturm–Liouville problems guarantees that the eigenfunctions form a complete set and that there are no nonreal eigenvalues. The only nontrivial question is whether there could be a negative or zero eigenvalue (corresponding to a pure imaginary ω). The easiest way to exclude this possibility (which you were not expected to do) is again to appeal to general theory: Assume X is an eigenfunction and Integrate by parts in

$$\omega^2 \int_0^\pi X(x)^2\,dx = -\int_0^\pi XX''\,dx = \beta X(\pi)^2 + \int_0^\pi X'(x)^2\,dx.$$

Since all the squares are positive, the assumption that ω^2 is strictly positive is vindicated.

The general Sturm–Liouville theorem guarantees that the eigenfunctions are complete and orthogonal, so the full solution for w has the form

$$w(t,x) = \sum_{j=1}^{\infty} c_j \cos(\omega_j x) e^{-\omega_j^2 t}.$$

$$h(x) = \sum_{j=1}^{\infty} c_j \cos(\omega_j x) \Longrightarrow c_j = \frac{\int_0^1 \cos(\omega_j x) h(x)\,dx}{\int_0^1 \cos^2(\omega_j x)\,dx}.$$

(C) Solve, by the method of your choice, the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(t,0) = 0 = u(t,1),$$
$$u(0,x) = f(x), \qquad \frac{\partial u}{\partial t}(0,x) = 0.$$

This can be solved either by separation of variables or by d'Alembert's method (with the odd extension of the initial data through each endpoint to enforce the Dirichlet boundary conditions). Please see the lecture notes for details.

(D) Solve the heat equation in a disk ($0 \le r < 1$),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$
$$u(t,1,\theta) = 0, \qquad u(0,r,\theta) = g(r)\cos\theta.$$

(The special choice of initial data assures that only one particular Bessel function appears in the answer; however, you will need to sum over something else.)

In general the solution of a heat problem in the disk involves a Fourier series in θ and a Bessel series in r. In the present case the Fourier series has only one term, proportional to $\cos\theta$, and we can set $\frac{\partial^2 u}{\partial\theta^2}$ equal to $-u$ from the start. So separation of variables, $u_{\text{sep}}(t,r,\theta) = T(t)R(r)\cos\theta$, yields

$$\frac{T'}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{1}{r^2} = -\omega^2.$$

The R equation is

$$R'' + \frac{1}{r}R' + \left(\omega^2 - \frac{1}{r^2}\right)R = 0,$$

whose solution (regular at $r = 0$) is $J_1(\omega r)$. Let ω_j $(j = 1, \ldots, \infty)$ be the values of ω for which $J_1(\omega) = 0$, so that the boundary condition is satisfied. Then the full solution must have the form

$$u(t,r,\theta) = \sum_{j=1}^{\infty} c_j J_1(\omega_j r)\cos\theta e^{-\omega_j^2 t}.$$

Because the eigenfunctions are orthogonal with weight factor r,

$$c_j = \frac{\int_0^1 J_1(\omega_j r) g(r)\, r\, dr}{\pi \int_0^1 J_1(\omega_j r)^2 r\, dr}.$$

(The π in the normalization factor comes from the integral of $\cos^2\theta$ over $(0, 2\pi)$.)