Math. 401, Sec. 500 Spring, 2005

Homework 11, due April 18

1. *[Schaum's, p. 94, Ex. 5.24]*

   (a) Find the Fourier transform of $f(x) = \begin{cases} \dfrac{1}{2\epsilon} & \text{if } |x| < \epsilon, \\ 0 & \text{if } |x| \geq \epsilon. \end{cases}$

   (b) Find the limit of this transform as $\epsilon \to 0^+$. Discuss the result.

2. *[Schaum's, p. 94, Ex. 5.26]* Find the Fourier sine transform and the Fourier cosine transform of

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1, \\ 0 & \text{if } x \geq 1. \end{cases}$$

3. *[Schaum's, p. 94, Ex. 5.27]*

   (a) Find the Fourier sine transform of e^{-x} $(x \geq 0)$.

   (b) Use (a) to show that for $m > 0$,

$$\int_0^\infty \frac{x \sin mx}{x^2 + 1}\, dx = \frac{\pi}{2}\, e^{-m}.$$

   (c) The formula in (b) fails when $m = 0$. Why does this not contradict the basic Fourier transform theorem? (See "Remarks" on p. 143 of Constanda, or "Convergence theorems" in the section "More on Fourier transforms" of the class notes, or "Fourier's integral theorem" on p. 80 of Schaum's.)

4. *[Schaum's, p. 94, Ex. 5.42]* An infinite thin bar $(-\infty < x < \infty)$ whose surface is insulated has an initial temperature

$$f(x) = \begin{cases} u_0 & \text{if } |x| < a, \\ 0 & \text{if } |x| \geq a. \end{cases}$$

   (a) Solve the heat equation $\dfrac{\partial u}{\partial t} = \kappa\, \dfrac{\partial^2 u}{\partial x^2}$ by separation of variables, obtaining a double integral.

   (b) Exchange the order of integration and evaluate the inner integral as the heat equation Green function (called "Gauss–Weierstrass kernel" by Constanda). Work the answer into the final form

$$u(x,t) = \frac{u_0}{2}\left[\operatorname{erf}\left(\frac{x+a}{2\sqrt{\kappa t}}\right) - \operatorname{erf}\left(\frac{x-a}{2\sqrt{\kappa t}}\right)\right].$$

   The *error function* erf is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du.$$

5. *[Schaum's, p. 94, Ex. 5.43]* A semiinfinite solid ($x > 0$) has initial temperature $f(x) = u_0 e^{-bx^2}$. The plane face, $x = 0$, is insulated. Show that the temperature is

$$u(x,t) = \frac{u_0}{\sqrt{1 + 4\kappa bt}}\, e^{-bx^2/(1+4\kappa bt)}.$$

(Follow the same two steps as in the previous problem.)

6. *[Schaum's, p. 94, Ex. 5.46]* Solve the potential problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for $y > 0$ with the boundary data

$$u(x,0) = \begin{cases} 0 & \text{if } x < -1 \text{ or } x > 1, \\ 1 & \text{if } -1 < x < 1. \end{cases}$$

7. *[Schaum's, p. 94, Ex. 5.49]* The lines $y = 0$ and $y = a$ in the xy plane are kept at potentials 0 and $f(x)$ respectively. In the strip between, the potential $u(x,y)$ obeys Laplace's equation (as in the previous problem). Show that the solution is

$$u(x,y) = \frac{1}{\pi} \int_{\lambda=0}^{\infty} \int_{\zeta=-\infty}^{\infty} f(\zeta)\, \frac{\sinh \lambda y}{\sinh \lambda a}\, \cos \lambda(\zeta - x)\, d\zeta\, d\lambda.$$