Math. 401, Sec. 500 Spring, 2005

Homework 14, due May 6, noon

This is an optional assignment for extra credit.

Turn in at most two of these problems. Do not do both problems marked "$*$".

1. $*$ Consider the Green function for the Laplace problem in the upper half plane,

$$G(x-z,y)=\frac{1}{\pi}\,\frac{y}{(x-z)^2+y^2}\,.$$

   (a) Verify that G satisfies Laplace's equation as a function of x and y.

   (b) Show that $\int_{-\infty}^{\infty} G(x-z,y)\,dz = 1$ for each fixed x and y.

   (c) Let $z=1$ and sketch $G(x-1,y)$ as a function of x for three representative values of y.

   (d) Justify the claim that

$$\lim_{y\to 0^+} G(x-z,y)=\delta(x-z).$$

   This means that for any well-behaved function f (say f continuous and bounded),

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty} G(x-z,y)\,f(z)\,dz = f(x).$$

   Hints: Write the integral as $\int_{-\infty}^{x-\delta}+\int_{x-\delta}^{x+\delta}+\int_{x+\delta}^{\infty}$, where δ is an arbitrarily small positive number. Show that the two outside integrals approach zero in the limit, using the assumption that f is bounded. In the middle integral, write $f(z)=f(x)+[f(z)-f(x)]$ and show that the integral involving the bracketed term can be assumed arbitrarily small, using the assumption that f is continuous.

   (e) Argue (a bit loosely, perhaps) from (a) and (d) that G is the correct Green function for the problem — without appeal to Fourier transforms or any other external information. (That is, show that

$$u(x,y)\equiv\int_{-\infty}^{\infty} G(x-z,y)\,f(z)\,dz$$

   is the bounded solution of Laplace's equation in the upper half plane with the boundary data $u(x,0)=f(x)$.)

2. Using the δ-function method, construct Green functions to solve these problems:

   (a) $$\frac{d^2y}{dx^2}+\omega^2 y = f(x),\qquad y'(0)=0,\quad y'(2)=0.$$
   (Assume $\omega>0$. For what values of ω does no solution exist?)

   (b) $$\frac{dy}{dt}-y=f(t),\qquad y(0)=0.$$
   Warning: In this case, G is *not* continuous at the location of the delta function.

3. $*$

   (a) Solve the wave equation in $\mathbf{R}^2$ by Fourier transforms, and rearrange the result into the form

   $$u(x,t) = \int_{-\infty}^{\infty} dy\,[V(x-y,t)f(y) + W(x-y,t)g(y)]\,,$$

   where f and g are the initial data (in the notation of the printed notes), and W and V are certain integral expressions.

   (b) What identities for the Fourier transforms of delta functions and step functions do you need to assume in order to get your answer to agree with the one on p. 136 of the notes? Try to boil it down to simple formulas for functions (or distributions) of *one* variable, not three.

4. Solve by the d'Alembert method (see pp. 136–139 of notes) the wave equation for $0 < x < \infty$ with Neumann boundary condition ($\frac{\partial u}{\partial x}(0,t) = 0$) and initial data

   $$u(x,0) \equiv f(x) = e^{-100(x-10)^2}, \qquad \frac{\partial u}{\partial t}(x,0) \equiv g(x) = 0.$$

   Sketch in space-time the pulses and the paths they follow. (The sketch should include at least the time interval $0 \le t \le 15$.)

5. Same as the previous problem, but with Dirichlet boundary condition ($u(0,t) = 0$) and initial data

   $$u(x,0) \equiv f(x) = 0, \qquad \frac{\partial u}{\partial t}(x,0) \equiv g(x) = x\,e^{-100(x-10)^2}.$$

6. Solve this problem in a quarter-disk ($0 < \theta < \frac{\pi}{2}$, $0 \le r < 1$):

   $$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial u}{\partial t}\,,$$

   $$u(t,r,0) = 0 = u(t,r,\pi/2), \qquad \frac{\partial u}{\partial r}(t,1,\theta) = 0,$$

   $$u(0,r,\theta) = g(r,\theta).$$