

Homework 3, due February 9

1. Find the first three terms (through order ϵ^2) of the Taylor series (in ϵ with t fixed) of

$$y(t) = \frac{e^{-\epsilon t}}{\sqrt{1 - \epsilon^2}} \sin(\sqrt{1 - \epsilon^2} t).$$

Compare with the perturbative solution found in class (or notes) for the damped oscillator equation. *Suggestion:* The easiest method is to find series for each of the three factors and then multiply them together like polynomials.

2. [Logan, p. 49, Ex. 1.3, beginning] Verify the following order relations:

- (a) $\epsilon^2 \tanh \epsilon = O(\epsilon^2)$ as $\epsilon \rightarrow \infty$.
- (b) $\exp(-\epsilon) = o(1)$ as $\epsilon \rightarrow \infty$.
- (c) $\sqrt{\epsilon(1 - \epsilon)} = O(\sqrt{\epsilon})$ as $\epsilon \rightarrow 0^+$.
- (d) $\frac{\sqrt{\epsilon}}{1 - \cos \epsilon} = O(\epsilon^{-3/2})$ as $\epsilon \rightarrow 0^+$.

3. [Logan, p. 49, Ex. 1.3, middle] Verify the following order relations:

- (a) $\epsilon = O(\epsilon^2)$ as $\epsilon \rightarrow \infty$.
- (b) $\exp(\epsilon) - 1 = O(\epsilon)$ as $\epsilon \rightarrow 0$.
- (c) $\int_0^\epsilon \exp(-x^2) dx = O(\epsilon)$ as $\epsilon \rightarrow 0^+$.
- (d) $\exp(\tan \epsilon) = O(1)$ as $\epsilon \rightarrow 0$.

4. [Logan, p. 49, Ex. 1.3, conclusion] Verify the following order relations:

- (a) $e^{-\epsilon} = O(\epsilon^{-p})$ as $\epsilon \rightarrow \infty$, for all $p > 0$.
- (b) $\ln \epsilon = o(\epsilon^{-p})$ as $\epsilon \rightarrow 0^+$, for all $p > 0$.

5. [Logan, p. 49, Ex. 1.16] Show that the three-term expansion

$$\sin t + \epsilon \cos t - \frac{\epsilon^2}{2} \sin t$$

is a uniformly valid approximation of $\sin(t + \epsilon)$ on $-\infty < t < \infty$. To justify your answer, look in a calculus book under “Taylor’s theorem with remainder”.

6. [Logan, p. 49, Ex. 1.15] Consider the boundary value problem (with $0 < \epsilon \ll 1$)

$$Ly \equiv t^2 y'' + \epsilon t^2 y' + \frac{1}{4}y = 0 \quad \text{for } 1 \leq t \leq e,$$

$$y(1) = 1, \quad y(e) = 0.$$

- (a) Use regular perturbation theory to find the leading-order behavior, $y_0(t)$. *Hint:* To solve the unperturbed equation, look in your differential equations textbook under “Euler equation” — or make a change of variable $t = e^u$.
- (b) Compute an upper bound for $|Ly_0|$ on $1 \leq t \leq e$ when $\epsilon = 0.01$.
7. [Logan, p. 49, Ex. 1.13] The equation of motion of a pendulum can be scaled to the form

$$\frac{d^2\theta}{d\tau^2} + \frac{\sin A\theta}{A} = 0,$$
$$\theta(0) = 1, \quad \frac{d\theta}{d\tau}(0) = 0$$

(where $\tau > 0$, $0 < A \ll 1$). Apply the regular perturbation method to find a two-term expansion. Show that the correction term is secular and comment on the validity of the approximation.