Math. 401, Sec. 500 Spring, 2005

Homework 5, due February 23

The first few problems here are a continuation of Problem 2 of the previous assignment, which concerned the equation

$$\frac{d^2y}{dt^2} + [\omega^2 - V(t)]y = 0. \tag{1}$$

1. (a) In the equation

$$\frac{d^2y}{dt^2} + \omega^2 V(t) y = 0 \tag{2}$$

make the change of time variable

$$\tau \equiv \omega \int_0^t \sqrt{V(\tilde{t})}\, d\tilde{t}.$$

Hint: As an intermediate step you should find that

$$\frac{d^2}{dt^2} = \omega^2 V(t) \frac{d^2}{d\tau^2} + \omega \frac{V'(t)}{2\sqrt{V(t)}} \frac{d}{d\tau}\,.$$

Show that the result is the equation of a harmonic oscillator subjected to small, slowly varying damping. (Define $\epsilon = 1/\omega$.)

(b) Apply the two-variable method to the damped oscillator equation you found in (a). (For help, see J. D. Cole, *Perturbation Methods in Applied Mathematics*, Sec. 3.6.) You should get the same WKB approximation

$$V(t)^{-\frac{1}{4}} e^{\pm i\omega \int_0^t \sqrt{V(\tilde{t})}\, d\tilde{t}} \tag{***}$$

that was obtained in class by applying the two-variable method directly to (2).

2. (a) In (2) make the change of variables

$$z \equiv V^{\frac{1}{4}} y, \qquad s \equiv \int_0^t \sqrt{V(\tilde{t})}\, d\tilde{t}.$$

[*Hint:* First carry out the change of dependent variable; then change the time variable in analogy with the previous problem.] Show that the result is a special case of (1).

(b) Problem 2(a) shows that equations (1) and (2) are, at root, equivalent. Therefore, the solution to Problem 2(b) of the previous assignment should somehow match the first-order WKB solution (derived in notes and in Problem 1). Check this, by renaming V in (2) as $1 - \epsilon^2 V$ and comparing the various solutions we've found.

3. The formula for the WKB approximation suggests that a solution which starts out proportional to $\exp(+i\omega \int \sqrt{V})$ remains so for all t (rather than acquiring a component proportional to $\exp(-i\omega \int \sqrt{V})$). This is actually true (through arbitrarily high finite order in $1/\omega$) if V is a smooth function. However, a discontinuity in V or its derivative can cause "mixing" of positive- and negative-frequency complex exponentials. To see this: For each of the following functions V, write down the WKB solution that behaves as $\exp(+i\omega \int \sqrt{V})$ on $-\infty < t < 0$. Match it to a WKB solution (linear combination of the two solutions (∗∗∗)) on $0 < t < \infty$ by requiring that the solution and its derivative be continuous at 0, at least to lowest order.

   (a) $$V(t) = \begin{cases} 1 & \text{for } t < 0, \\ 2 & \text{for } t > 0. \end{cases}$$

   (b) $$V(t) = \begin{cases} 1 & \text{for } t < 0, \\ 1+t & \text{for } t > 0. \end{cases}$$

   Comment on the difference between the two cases. (Relate the asymptotic dependence on ω to the degree of smoothness (continuity and differentiability) of V.)

4. Consider the (second-order, linear) partial differential equation

$$i\epsilon \frac{\partial u}{\partial t} = -\frac{\epsilon^2}{2m}\nabla^2 u + V(\mathbf{x})u. \tag{S}$$

   Here $\mathbf{x}$ is a variable in $\mathbf{R}^3$ and ∇ is the gradient with respect to $\mathbf{x}$; $u(t,\mathbf{x})$ thus is a complex-valued function of four real variables. Make the ansatz

$$u(t,\mathbf{x}) = e^{iS(t,\mathbf{x})/\epsilon}\left[A_0(t,\mathbf{x}) + \epsilon A_0(t,\mathbf{x}) + \cdots\right]$$

   (where S and A_n are independent of ϵ). Show that the first two equations in the perturbation hierarchy are

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V = 0 \tag{HJ}$$

   and

$$\frac{\partial A_0}{\partial t} + \frac{1}{m}\nabla A_0 \cdot \nabla S + \frac{1}{2m} A_0 \nabla^2 S = 0. \tag{T}$$

   Remarks: In quantum mechanics (S) is the *Schrödinger equation* describing the behavior of a particle of mass m; ϵ is *Planck's constant*, usually written as $\hbar$. Then (HJ) (a first-order, nonlinear equation for S) is called the *Hamilton–Jacobi equation*, and in advanced classical mechanics courses it is shown that solving it is equivalent to solving the classical (nonquantum) equation of motion of the particle. Finally, (T) (a first-order, linear equation for A_0) is called the *transport equation* because it can be solved by integration along the possible classical trajectories of the particle.