

Homework 8, due March 23

1. [Logan, p. 195, Ex. 3.1(d)] Find the eigenvalues and eigenfunctions for the problem

$$y'' + \lambda y = 0 \quad \text{on } 0 < x < L,$$

$$y'(0) = 0, \quad y(L) = 0.$$

2. Suppose that the boundary conditions (“BC:”) on p. 67 of the notes are replaced by

$$\frac{\partial w}{\partial x}(t, 0) = 0, \quad \frac{\partial w}{\partial x}(t, 1) = 0.$$

What changes would be necessary in pp. 67–72 (and corresponding lecture on separation of variables in the heat equation)? *Warning:* Pay close attention to the possibility that λ may not always be positive.

3. [Logan, p. 195, Ex. 3.4] Find the Fourier sine series of $f(x) = 1$ on $0 \leq x \leq \pi$.
4. [Logan, p. 195, Ex. 3.9] Solve by separation of variables (getting a Fourier sine series):

$$\begin{aligned} u_{xx} + u_{yy} &= 0 && \text{on } 0 < x < a, \quad 0 < y < b, \\ u(x, 0) &= 0 && \text{and } u(x, b) = f(x) \quad \text{for } 0 < x < a, \\ u(0, y) &= u(a, y) = 0 && \text{for } 0 < y < b. \end{aligned}$$

5. [Schaum's, p. 46, Ex. 2.43]

- (a) Show that for $-\pi < x < \pi$,

$$x = 2 \left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right).$$

- (b) By integrating the result of (a), show that for $-\pi \leq x \leq \pi$,

$$x^2 = \frac{\pi^2}{3} - 4 \left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots \right).$$

(Determine the constant of integration by requiring that the series formula give the right answer for $\int_0^\pi x^2 dx$.)

- (c) By integrating once more and being careful about constants of integration, show that

$$x(\pi - x)(\pi + x) = 12 \left(\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \dots \right).$$

6. [Schaum's, p. 46, Ex. 2.37]

(a) Expand $f(x) = \cos x$, $0 < x < \pi$, in a Fourier sine series.

(b) How should $f(x)$ be defined at $x = 0$ and $x = \pi$ so that the series will converge to $f(x)$ for $0 \leq x \leq \pi$?

7. [Schaum's, p. 46, Ex. 2.50(a)] Solve the boundary-value problem

$$\frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2}, \quad u(0, t) = u(4, t) = 0, \quad u(x, 0) = 25x.$$

It is understood that $0 < x < 4$ and $t > 0$.