Math. 401 (Fulling) 25 February 2005

Test A – Solutions

“Up **to** [a certain order]” *means* “stop right before calculating the term of that order.”
“Up **through** [a certain order]” *means* “**do** calculate a term of that order.”

Calculators may be used for simple arithmetic operations only!

1. *(30 pts.)* Here are three problems, each of which requires a different flavor of singular perturbation theory. For each problem, tell which method you would expect to work. For full credit, *state the ansatz in as much detail as you can*, but don't try to solve for the unknown elements (because you won't have time).

(a) $$\frac{d^2y}{dt^2} + \omega^2(100+t^2)y = 0 \qquad (\omega \to +\infty).$$

The WKB approximation applies. The general solution is a linear combination of the two solutions approximated by

$$(100+t^2)^{-1/4} e^{\pm i\omega \int^t \sqrt{100+\tilde{t}^2}\, d\tilde{t}}.$$

Because $100 + t^2$ is always positive, this approximation is valid for all t.

(b) $$\frac{d^2y}{dt^2} + 36y + \epsilon y^5 = 0 \qquad (\epsilon \to 0).$$

The distorted-time (Poincaré) method is the best approach to a nonlinear oscillator. Change variable to $\tau = t + \epsilon\omega_1 t + \epsilon^2\omega_2 t + \cdots$, where the ω_n are constants in this case because there are no t-dependent coefficients in the differential equation. Then let $y = y_0(\tau) + \epsilon y_1(\tau) + \cdots$. When solving for y_n, choose ω_n to eliminate any resonant forcing terms (proportional to $\cos(6\tau)$ or $\sin(6\tau)$ in this case).

(c) $$\frac{d^2y}{dt^2} + \epsilon\left(\frac{dy}{dt}\right)^3 + 9y = 0 \qquad (\epsilon \to 0^+).$$

The damping suggests that a two-time ansatz is necessary. Let $u = \epsilon t$ and $y = y_0(t,u) + \epsilon y_1(t,u) + \cdots$. The solution for y_0 will involve “constants” of integration that depend on u and are not completely determined by the zeroth-order problem. To get first-order differential equations for these functions of u one needs to look at the first-order problem and require that no resonant forcing terms ($\cos(3\tau)$ or $\sin(3\tau)$) appear.

2. *(35 pts.)* Consider $\epsilon x^4 - x^2 + 3x - 2 + \epsilon = 0 \quad (\epsilon \to 0)$.

(a) Find the leading behavior of all roots.

First let's try regular perturbation theory, just setting $\epsilon = 0$ to get an unperturbed problem:

$$0 = -x^2 + 3x - 2 = -(x-2)(x-1).$$

Therefore, we expect two roots with the behavior

$$x = 1 + \cdots \qquad \text{and} \qquad x = 2 + \cdots .$$

Since this is a fourth-order equation, there should be two more roots. Balancing the x^4 term with the largest ϵ-independent term, x^2, we expect that $\epsilon x^2 \sim 1$, or $x \sim \epsilon^{-1/2}$. Therefore, define $\overline{x}$ by $x = \epsilon^{-1/2}\overline{x}$ and substitute into the equation, getting

$$\epsilon^{-1}\overline{x}^4 - \epsilon^{-1}\overline{x}^2 + 3\epsilon^{-1/2}\overline{x} - 2 + \epsilon = 0.$$

Multiply by ϵ:

$$\overline{x}^4 - \overline{x}^2 + 3\epsilon^{+1/2}\overline{x} - 2\epsilon + \epsilon^2 = 0.$$

In the lowest order we need to solve

$$0 = \overline{x}^4 - \overline{x}^2 = \overline{x}^2(\overline{x}^2 - 1) = \overline{x}^2(\overline{x} - 1)(\overline{x} + 1).$$

The two zero roots will simply reproduce the two solutions we found earlier. The others are $\overline{x} = \pm 1$, or

$$x = \epsilon^{-1/2} + \cdots \qquad \text{and} \qquad x = -\epsilon^{-1/2} + \cdots.$$

(b) For the smallest root, find a perturbative solution through order ϵ . (More precisely, "smallest root" means "the real root that is smallest in the limit $\epsilon \to 0^+$.")

The root in question is the one with $x_0 = 1$. So we set $x \sim 1 + \epsilon x_1$ and calculate

$$x^4 = 1 + O(\epsilon)\,, \quad x^2 = 1 + 2\epsilon x_1 + O(\epsilon^2)\,,$$

so

$$0 = \epsilon + (-1 - 2\epsilon x_1) + (3 + 3\epsilon x_1) - 2 + \epsilon + O(\epsilon^2).$$

The zeroth-order terms, $-1+3-2$, cancel of course, since we already solved the zeroth-order equation to get $x_0 = 1$. The equation of order ϵ is

$$0 = 1 - 2x_1 + 3x_1 + 1 = 2 + x_1\,,$$

whence $x_1 = -2$. Finally, therefore,

$$x = 1 - 2\epsilon + O(\epsilon^2)\,.$$

3. *(35 pts.)* Let's treat by **regular** perturbation theory (as far as it will take us) the problem

$$\frac{d^2y}{dt^2} + \epsilon\left(\frac{dy}{dt}\right)^2 + y = 0, \qquad y(0) = 0, \quad \frac{dy}{dt}(0) = 1.$$

(a) Find the first-order solution (i.e., through order ϵ).

Since we'll need it in part (b), let's install the full second-order ansatz, $y \sim y_0 + \epsilon y_1 + \epsilon^2 y_2$.

$$y'' = y_0'' + \epsilon y_1'' + \epsilon^2 y_2'' + O(\epsilon^3), \quad (y')^2 = (y_0' + \epsilon y_1' + O(\epsilon^2))^2 = (y_0')^2 + 2\epsilon y_0' y_1' + O(\epsilon^2).$$

Thus

$$0 = y_0'' + \epsilon y_1'' + \epsilon^2 y_2'' + \epsilon (y_0')^2 + 2\epsilon^2 y_0' y_1' + y_0 + \epsilon y_1 + \epsilon^2 y_2 + O(\epsilon^3).$$

The expansion of the initial conditions is trivial: $y_0'(0) = 1$ and all the other initial values are 0.

The zeroth-order problem is

$$y_0'' + y_0 = 0, \qquad y_0(0) = 0, \quad y_0'(0) = 1.$$

The solution is $y_0(t) = \sin t$.

The first-order problem is

$$y_1'' + y_1 = -(y_0')^2 = -\cos^2 t = -\tfrac{1}{2} - \tfrac{1}{2}\cos(2t)$$

with $y_1(0) = 0 = y_1'(0)$. There are no resonant terms, so *regular perturbation theory is adequate through this order.* We look for a particular solution of the form

$$y_{\rm p} = A\cos(2t) + B\sin(2t) + C.$$

Substituting, we get

$$-4A\cos(2t) - 4B\sin(2t) + A\cos(2t) + B\sin(2t) + C = -\tfrac{1}{2} - \tfrac{1}{2}\cos(2t),$$

whence $C = -\frac{1}{2}$, $B = 0$, $A = \frac{1}{6}$. Thus

$$y_1 = c_1\cos t + c_2\sin t + \tfrac{1}{6}\cos(2t) - \tfrac{1}{2}\,.$$

Imposing the initial conditions gives

$$0 = c_1 - \tfrac{1}{3}\,, \qquad 0 = c_2\,,$$

or

$$y(t) \sim \sin t + \epsilon\left[\tfrac{1}{3}\cos t + \tfrac{1}{6}\cos(2t) - \tfrac{1}{2}\right].$$

(b) Carry the calculation of the second-order term ($\epsilon^2 y_2$) far enough to demonstrate that it will contain a secular term (although y_1 does not). Earn extra credit by doing something more, if you have time. (You have some freedom of judgment in deciding what "more" to do.)

Continuing from above we get the second-order problem

$$\begin{aligned} y_2'' + y_2 &= -2y_0'y_1' \\ &= -2\cos t[-\tfrac{1}{3}\sin t - \tfrac{1}{3}\sin(2t)] \\ &= \tfrac{1}{3}\sin(2t) + \tfrac{1}{3}[\sin(3t) + \sin t] \end{aligned}$$

with $y_2(0) = 0 = y_2'(0)$. The $\sin t$ term is resonant, so it will produce a secular term in y_2.

Helpful formulas:

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x), \qquad \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

$$\sin x\cos y = \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)$$

$$\cos x\cos y = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)$$

$$\sin x\sin y = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$$