Math. 401 (Fulling) 11 April 2005

Test B – Solutions

Calculators may be used for simple arithmetic operations only!

1. *(30 pts.)* Find the lowest-order outer, inner, and uniform solutions for

$$\epsilon \frac{d^2y}{dt^2} + \frac{dy}{dt} + 4y^2 = 0\,, \quad 0 < t < 1\,, \quad 0 < \epsilon \ll 1\,,$$

with the boundary conditions $y(0) = 0$, $y(1) = 1$.

Since ϵ and the first-derivative coefficient have the same sign, the boundary layer is on the left.

Outer solution: With $\epsilon = 0$ we have $y_{\rm o}' + 4{y_{\rm o}}^2 = 0$, hence

$$\int \frac{dy_{\rm o}}{{y_{\rm o}}^2} = -4\,dt \qquad \Rightarrow \qquad -\frac{1}{y_{\rm o}} = -4t + C \qquad \Rightarrow \qquad y_{\rm o} = \frac{1}{4t - C}\,.$$

Since $y_{\rm o}(1) = 1$, we have $4 - C = 1$, or

$$y_{\rm o}(t) = \frac{1}{4t-3}\,.$$

Inner solution: Let $\tau = t/\epsilon$, so $t = \epsilon\tau$ and $\frac{d}{dt} - \frac{1}{\epsilon}\frac{d}{d\tau}$. Then the ODE transforms to

$$\frac{1}{\epsilon}\frac{d^2y}{d\tau^2} + \frac{1}{\epsilon}\frac{dy}{d\tau} + 4y^2 = 0 \qquad \Rightarrow \qquad \frac{d^2y_{\rm i}}{d\tau^2} + \frac{dy_{\rm i}}{d\tau} = O(\epsilon).$$

Thus to zeroth order

$$\frac{dy_{\rm i}}{d\tau} = Ae^{-\tau} \qquad \Rightarrow \qquad y_{\rm i} = -Ae^{-\tau} + B.$$

Since $y_{\rm i}(0) = 0$, we get $B = A$.

To determine A we need to look on an intermediate scale, $\eta = \frac{t}{\sqrt{\epsilon}} = \sqrt{\epsilon}\,\tau$. Then as $\epsilon \to 0$ we have

$$y_{\rm i} = A(1 - e^{-\eta/\sqrt{\epsilon}}) \to A, \qquad y_{\rm o} = \frac{1}{4\sqrt{\epsilon}\,\eta - 3} \to -\frac{1}{3}\,,$$

so $A = -\frac{1}{3}$ for consistency. Thus

$$y_{\rm i}(\tau) = \frac{1}{3}(e^{-\tau} - 1).$$

Uniform (composite) solution: $y \sim y_{\rm o} + y_{\rm i} - (\text{common limit}) =$

$$\frac{1}{4t-3} + \frac{1}{3}(e^{-t/\epsilon} - 1) + \frac{1}{3} = \frac{1}{4t-3} + \frac{1}{3}\,e^{-t/\epsilon}.$$

Check:

$$y(0) = 0. \quad \surd \qquad\qquad y(1) = 1 + O(e^{-1/\epsilon}) \sim 1. \quad \surd$$

$$\frac{dy}{dt} = \frac{-4}{(4t-3)^2} - \frac{1}{3\epsilon}\, e^{-t/\epsilon},$$

$$\frac{d^2y}{dt^2} = \frac{32}{(4t-3)^3} + \frac{1}{3\epsilon^2}\, e^{-t/\epsilon},$$

$$4y^2 = \frac{4}{(4t-3)^2} + \frac{8}{3}\,\frac{1}{4t-3}\, e^{-t/\epsilon} + \frac{1}{9}\, e^{-2t/\epsilon}.$$

Adding up, we see that the terms of the leading orders, $O(\epsilon^{-1}e^{-t/\epsilon})$ and $O(\epsilon^0)$, cancel, leaving

$$\epsilon\,\frac{d^2y}{dt^2} + \frac{dy}{dt} + 4y^2 = \frac{32\epsilon}{(4t-3)^3} + \frac{8}{3}\,\frac{1}{4t-3}\, e^{-t/\epsilon} + \frac{1}{9}\, e^{-2t/\epsilon} = O(\epsilon^0 e^{-t/\epsilon}) + O(\epsilon).$$

This is the best degree of cancellation that can be expected when the outer and inner solutions are constructed only through zeroth order.

2. *(20 pts.)* Suppose that f is a function on the interval $0 < x < \pi$ whose graph resembles

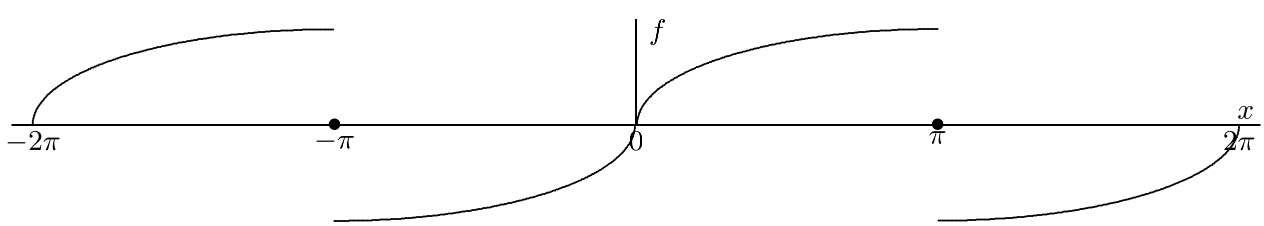
Then we know that f has a Fourier sine series of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)\,.$$

Furthermore, this series represents a periodic function defined on the whole real line, which we also call f.

(a) Sketch the graph of f over the interval $-2\pi \le x \le 2\pi$.

The odd periodic extension:



(The dots at $x = \pm\pi$ are part of the graph.)

(b) Over the same interval as in (a), sketch the graph of a typical partial sum, such as

$$f_{10}(x) = \sum_{n=1}^{10} b_n \sin(nx)\,.$$

(If you use the same axes for (a) and (b), please use a different color of pen or pencil.)
[See separate Maple output for an exact plot.]

(c) Does the series converge to f "uniformly"?
NO — the function is discontinuous, and the convergence can't be uniform near the jump.

(d) Does the series converge to f "in the mean"?
YES — the function is square-integrable.

In answering the remaining questions it is not necessary to rederive "well known" facts about Fourier series; just use them, making sure that your reasoning is clear. (For example, you are not expected to prove that there are no negative eigenvalues.)

3. *(30 pts.)* Solve Laplace's equation, $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$, on the unit square $0 < x < 1$, $0 < y < 1$, with the boundary data

$$\frac{\partial v}{\partial x}(0,y) = 0 = \frac{\partial v}{\partial x}(1,y)\,, \qquad v(x,0) = 0\,, \quad v(x,1) = f(x)\,.$$

Separation step: Try $v_{\text{sep}} = X(x)Y(y)$. Then

$$X''Y + XY'' = 0 \qquad \Rightarrow \qquad \frac{X''}{X} = -\frac{Y''}{Y} = -k^2.$$

The boundary conditions $X'(0) = 0 = X'(1)$ imply that

$$X(x) = \cos(kx) \quad \text{with } 0 = -k\sin(k)\,, \qquad \text{hence } k = n\pi\,,\ n = 0\,,1\,,\ \ldots.$$

(In particular, this justifies the assumption that $-k^2$ is a nonpositive number.) The other equation, $Y'' = k^2 Y$ with $Y(0) = 0$, then gives

$$Y(y) = \begin{cases} \sinh(n\pi y) & \text{if } n \neq 0, \\ y & \text{if } n = 0 \end{cases}$$

(times arbitrary constants).

Superposition step: Write

$$v(x,y) = a_0 y + \sum_{n=1}^{\infty} a_n \cos(n\pi x)\sinh(n\pi y).$$

Then

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)\sinh(n\pi).$$

Therefore,

$$a_0 = \int_0^1 f(x)\,dx,$$

$$a_n = \frac{2}{\sinh(n\pi)} \int_0^1 \cos(n\pi x) f(x)\,dx \quad \text{if } n > 0\,.$$

4. *(Essay – 20 pts.)* Outline a strategy for solving the heat equation on the unit square $(\,0 < x < 1\,,\ 0 < y < 1\,,\ 0 < t < \infty\,)$,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\,, \qquad u(0,x,y) = g(x,y)\,,$$

$$\frac{\partial u}{\partial x}(t,0,y) = 0 = \frac{\partial u}{\partial x}(t,1,y)\,,$$

$$u(t,x,0) = 0\,, \quad u(t,x,1) = f(x) \text{ (independent of } t\,).$$

For **extra credit**, carry out the solution as far as you can in the available time.

Let $w = u - v$, where v is the steady-state solution found in the previous problem. Then

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\,, \qquad w(0,x,y) = g(x,y) - v(x,y)\,,$$

$$\frac{\partial w}{\partial x}(t,0,y) = 0 = \frac{\partial w}{\partial x}(t,1,y)\,,$$

$$w(t,x,0) = 0 = w(t,x,1)\,.$$

This problem can be solved by separation of variables. Look for normal modes of the form

$$T(t)X(x)Y(y).$$

We will get $X(x) = \cos(n\pi x)$ as in Question 3, and $Y(y) = \sin(m\pi y)$. The final solution will be a superposition involving a double sum over the indices m and n. The formulas for the coefficients will be integrals (over the square) of $g(x,y) - v(x,y)$ times the eigenfunctions $X_n(x)Y_m(y)$.

That should be enough for 20 points. Since the rest is extra credit, I don't have to type it out if I don't have time.