Math. 401, Sec. 501 Spring, 2006

Homework 13, due May 2

Everybody should do the first problem, but not turn it in. The rest is an optional assignment for extra credit; turn in at most two of the problems.

1. *[Logan, p. 159, Ex. 1.3]* Determine in which regions each equation is hyperbolic, elliptic, or parabolic. Subscripts indicate partial differentiation with respect to the indicated variables.

   (a) $t\,u_{tt} + u_{xx} = 0$

   (b) $u_{tt} - u_{xx} = 0$

   (c) $u_{tt} + (1+x^2)u_x - u_t = e^t$

   (d) $u_{xx} + u_{yy} = f(x,y)$

2. Let r and θ be polar coordinates, defined in the usual way:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Show that

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\,.$$

Hint: First show that (acting on any function)

$$\frac{\partial}{\partial x} = \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial \theta}\,,$$

$$\frac{\partial}{\partial y} = \sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial \theta}\,.$$

Hint for hint: The calculations are easier if you use some kind of implicit differentiation. For example, a quick way to show that

$$\frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}$$

is to differentiate the equation

$$\tan\theta = \frac{y}{x}$$

with respect to y (with x fixed) and perform some necessary algebra on the result. Remember that (for example) a y derivative with x fixed is not the same as a y derivative with r fixed!

3. *[Logan, p. 304, Ex. 3.13]* In three dimensions the wave equation is

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0$$

where ∇^2 is the Laplacian. For waves with spherical symmetry, $u = u(r,t)$ and

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\,.$$

By introducing the variable $U = ru$, show that the general solution for the spherically symmetric wave equation is

$$u = \frac{1}{r} f(r - ct) + \frac{1}{r} g(r + ct).$$

4. (a) Solve the wave equation in $\mathbf{R}^2$ by Fourier transforms, and rearrange the result into the form

$$u(x,t) = \int_{-\infty}^{\infty} dy\,[V(x-y,t)f(y) + W(x-y,t)g(y)]\,,$$

where f and g are the initial data (in the notation of the printed notes), and W and V are certain integral expressions.

(b) What identities for the Fourier transforms of delta functions and step functions do you need to assume in order to get your answer to agree with the one on p. 137 of the notes? Try to boil it down to simple formulas for functions (or distributions) of *one* variable, not three.

5. Solve by the d'Alembert method (see pp. 137–140 of notes) the wave equation for $0 < x < \infty$ with Neumann boundary condition ($\frac{\partial u}{\partial x}(0,t) = 0$) and initial data

$$u(x,0) \equiv f(x) = e^{-100(x-10)^2}, \qquad \frac{\partial u}{\partial t}(x,0) \equiv g(x) = 0.$$

Sketch in space-time the pulses and the paths they follow. (The sketch should include at least the time interval $0 \le t \le 15$.)

6. Same as the previous problem, but with Dirichlet boundary condition ($u(0,t) = 0$) and initial data

$$u(x,0) \equiv f(x) = 0, \qquad \frac{\partial u}{\partial t}(x,0) \equiv g(x) = x\,e^{-100(x-10)^2}.$$

7. Solve this problem in a quarter-disk ($0 < \theta < \frac{\pi}{2}$, $0 \le r < 1$):

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial u}{\partial t}\,,$$

$$u(t,r,0) = 0 = u(t,r,\pi/2), \qquad \frac{\partial u}{\partial r}(t,1,\theta) = 0,$$

$$u(0,r,\theta) = g(r,\theta).$$