Math. 401, Sec. 501 Spring, 2006

Homework 3, due February 8

1. Find the first three terms (through order ϵ^2) of the Taylor series (in ϵ with t fixed) of

$$y(t) = \frac{e^{-\epsilon t}}{\sqrt{1-\epsilon^2}} \sin(\sqrt{1-\epsilon^2}\, t).$$

   Compare with the perturbative solution found in class (or notes) for the damped oscillator equation. *Suggestion:* The easiest method is to find series for each of the three factors and then multiply them together like polynomials.

2. *[Logan, p. 49, Ex. 1.3, beginning]* Verify the following order relations:

   (a) $\epsilon^2 \tanh \epsilon = O(\epsilon^2)$ as $\epsilon \to \infty$.

   (b) $\exp(-\epsilon) = o(1)$ as $\epsilon \to \infty$.

   (c) $\sqrt{\epsilon(1-\epsilon)} = O(\sqrt{\epsilon})$ as $\epsilon \to 0^+$.

   (d) $\dfrac{\sqrt{\epsilon}}{1-\cos\epsilon} = O(\epsilon^{-3/2})$ as $\epsilon \to 0^+$.

3. *[Logan, p. 49, Ex. 1.3, middle]* Verify the following order relations:

   (a) $\epsilon = O(\epsilon^2)$ as $\epsilon \to \infty$.

   (b) $\exp(\epsilon) - 1 = O(\epsilon)$ as $\epsilon \to 0$.

   (c) $\int_0^\epsilon \exp(-x^2)\, dx = O(\epsilon)$ as $\epsilon \to 0^+$.

   (d) $\exp(\tan\epsilon) = O(1)$ as $\epsilon \to 0$.

4. *[Logan, p. 49, Ex. 1.3, conclusion]* Verify the following order relations:

   (a) $e^{-\epsilon} = O(\epsilon^{-p})$ as $\epsilon \to \infty$, for all $p > 0$.

   (b) $\ln\epsilon = o(\epsilon^{-p})$ as $\epsilon \to 0^+$, for all $p > 0$.

5. *[Logan, p. 49, Ex. 1.16]* Show that the three-term expansion

$$\sin t + \epsilon \cos t - \frac{\epsilon^2}{2} \sin t$$

   is a uniformly valid approximation of $\sin(t+\epsilon)$ on $-\infty < t < \infty$. To justify your answer, look in a calculus book under "Taylor's theorem with remainder".

6. *[Logan, p. 49, Ex. 1.15]* Consider the boundary value problem (with $0 < \epsilon \ll 1$)

$$Ly \equiv t^2 y'' + \epsilon t^2 y' + \tfrac{1}{4} y = 0 \quad \text{for } 1 \le t \le e,$$

$$y(1) = 1, \qquad y(e) = 0.$$

(a) Use regular perturbation theory to find the leading-order behavior, $y_0(t)$. *Hint:* To solve the unperturbed equation, look in your differential equations textbook under "Euler equation" — or make a change of variable $t = e^u$.

(b) Compute an upper bound for $|Ly_0|$ on $1 \le t \le e$ when $\epsilon = 0.01$.

7. *[Logan, p. 49, Ex. 1.13]* The equation of motion of a pendulum can be scaled to the form

$$\frac{d^2\theta}{dt^2} + \frac{\sin A\theta}{A} = 0,$$

$$\theta(0) = 1, \qquad \frac{d\theta}{dt}(0) = 0$$

(where $t > 0$ and $0 < A \ll 1$). Apply the regular perturbation method to find a two-term expansion. (Define $\epsilon = A^2$.) Show that the second term (the one of order $O(A^2)$) is secular and comment on the validity of the approximation.