

Homework 9, due March 29

1. [Logan, p. 195, Ex. 3.2–3]

(a) Let $f(x) = x^2$ on $-\pi \leq x \leq \pi$ and $f(x + 2\pi) = f(x)$ for all x . Show that

$$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos(nx)}{n^2}.$$

(b) Use this series to prove that $\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$.

2. [Logan, p. 195, Ex. 3.6] Find the (“full”, exponential) Fourier series for the periodic function defined by

$$f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0, \\ 1 & \text{for } 0 \leq x < \pi. \end{cases}$$

Compare with the Fourier sine series you found for this function last week (Exercise 2 of Homework 8.) Explain the differences (if any) between them.

3. [Schaum’s, p. 46, Ex. 2.34–5(b)] If

$$f(x) = \begin{cases} -x & \text{when } -4 \leq x \leq 0, \\ x & \text{when } 0 \leq x \leq 4, \end{cases}$$

and f is periodic with period 8, graph the function and find its Fourier series (using properties of even or odd functions whenever applicable). Also, tell where the discontinuities of f are located and to what value the series converges at each discontinuity.

4. [Schaum’s, p. 46, Ex. 2.34–5(c)] If $f(x) = 4x$ for $0 < x < 10$ and f is periodic with period 10 (note: **not** 20), graph the function and find its Fourier series. Also, tell where the discontinuities of f are located and to what value the series converges at each discontinuity.

5. [Schaum’s, p. 46, Ex. 2.38]

(a) Expand $f(x) = \cos x$, $0 < x < \pi$, in a (“full”) Fourier series.

(b) Expand $f(x) = \cos x$, $0 < x < \pi$, in a Fourier cosine series.

(c) Compare these results with the Fourier sine series you found for this function last week (Exercise 5 of Homework 8). Explain the differences (if any) among them.

6. [Schaum's, p. 46, Ex. 2.40] Show that for $0 \leq x \leq \pi$,

$$(a) \quad x(\pi - x) = \frac{\pi^2}{6} - \left(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \cdots \right).$$

$$(b) \quad x(\pi - x) = \frac{8}{\pi} \left(\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots \right).$$

7. [Schaum's, p. 46, Ex. 2.41] Use the results of Exercise 6 to show (by judicious choices of x) that

$$(a) \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$$(b) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

$$(c) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32}.$$

8. [Schaum's, p. 46, Ex. 2.46 and 2.48] Use Parseval's identity along with Exercise 4 of Homework 8 and Exercise 6 of this assignment to show (in any convenient order)

$$(a) \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

$$(b) \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

$$(c) \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}.$$

$$(d) \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}.$$